commutativity of the tensor product. ∎

We can now show that the torsion product of A and B depends only on the torsion submodules of A and B.

11 COROLLARY *Let A and B be modules and let i: $\operatorname{Tor} A \subset A$ and j: $\operatorname{Tor} B \subset B$. Then $i * j$: $\operatorname{Tor} A * \operatorname{Tor} B \approx A * B$.*

PROOF There is a short exact sequence

$$0 \to \operatorname{Tor} B \xrightarrow{j} B \to B/\operatorname{Tor} B \to 0$$

where $B/\operatorname{Tor} B$ is without torsion. By lemma 5, $A * (B/\operatorname{Tor} B) = 0$, and, by corollary 9, $1 * j$: $A * \operatorname{Tor} B \approx A * B$. By a similar argument, there is an isomorphism $i * 1$: $\operatorname{Tor} A * \operatorname{Tor} B \approx A * \operatorname{Tor} B$, and the composite of these gives the result. ∎

We use these results to extend the universal-coefficient theorem. Given a chain complex C over R, a *free approximation of* C is a chain map τ: $\bar{C} \to C$ such that

(*a*) $\bar{C}$ is a free chain complex over R.
(*b*) τ is an epimorphism.
(*c*) τ induces an isomorphism $\tau_*: H(\bar{C}) \approx H(C)$.

12 LEMMA *Any chain complex C has a free approximation, uniquely determined up to homotopy equivalence.*

PROOF For each $q \geq 0$ choose a homomorphism α_q: $F_q \to Z_q(C)$ such that F_q is a free R module and α_q is an epimorphism. Let $F'_q = \alpha_q{}^{-1}(B_q(C))$ and choose a homomorphism β_q: $F'_q \to C_{q+1}$ such that $\partial_{q+1}\beta_q = \alpha_q \mid F'_q$ [such a homomorphism exists because F'_q is free and ∂_{q+1}: $C_{q+1} \to B_q(C)$ is an epimorphism]. Define $\bar{C}_q = F_q \oplus F'_{q-1}$ and define homomorphisms

$$\bar{\partial}_q\colon \bar{C}_q \to \bar{C}_{q-1} \qquad \text{by} \qquad \bar{\partial}_q(a,b) = (b,0)$$
$$\tau_q\colon \bar{C}_q \to C_q \qquad \text{by} \qquad \tau_q(a,b) = \alpha_q(a) + \beta_{q-1}(b)$$

Then $\bar{C} = \{\bar{C}_q,\bar{\partial}_q\}$ is a free chain complex and $\tau = \{\tau_q\}$ is a chain map from $\bar{C}$ to C. τ is epimorphic because $\tau_q(\bar{C}_q) \supset \ker \partial_q$ and $\partial_q\tau_q(\bar{C}_q) \supset \operatorname{im} \partial_q$. Since $Z_q(\bar{C}) = F_q$, $B_q(\bar{C}) = F'_q$, and $\tau_q(Z_q(\bar{C})) = \alpha_q(F_q)$, it follows that

$$\tau_*: Z_q(\bar{C})/B_q(\bar{C}) \approx Z_q(C)/B_q(C)$$

Therefore τ: $\bar{C} \to C$ is a free approximation of C. The uniqueness will follow from lemma 13 below. ∎

If τ: $\bar{C} \to C$ is a free approximation of C, there is a subcomplex $\bar{\bar{C}} = \{\bar{\bar{C}}_q = \ker \tau_q\colon \bar{C}_q \to C_q\}$ of $\bar{C}$ and a short exact sequence of chain complexes

$$0 \to \bar{\bar{C}} \xrightarrow{i} \bar{C} \xrightarrow{\tau} C \to 0$$

Because $\tau_*: H(\bar{C}) \approx H(C)$, it follows from the exactness of the homology sequence of the above short exact sequence that $\bar{\bar{C}}$ is acyclic (see corollary 4.5.5a). Since $\bar{\bar{C}}$ is a free chain complex (because it is a subcomplex of a free chain complex), it follows from theorem 4.2.5 that $\bar{\bar{C}}$ is contractible. We use this in the following lemma.

13 LEMMA *Given a free approximation τ: $\bar{C} \to C$ of C and given a free chain complex C' and a chain map τ': $C' \to C$, there exist chain maps $\bar{\tau}$: $C' \to \bar{C}$ such that $\tau \circ \bar{\tau} = \tau'$, and any two are chain homotopic.*

PROOF As above, there is a short exact sequence of chain complexes

$$0 \to \bar{\bar{C}} \xrightarrow{i} \bar{C} \xrightarrow{\tau} C \to 0$$

where $\bar{\bar{C}}$ is chain contractible. Let $D = \{D_q\colon \bar{\bar{C}}_q \to \bar{\bar{C}}_{q+1}\}$ be a contraction of $\bar{\bar{C}}$. Because C'_q is free and τ_q: $\bar{C}_q \to C_q$ is an epimorphism, there is a homomorphism φ_q: $C'_q \to \bar{C}_q$ such that $\tau_q \varphi_q = \tau'_q$. Then

$$h_q = \bar{\partial}_q \varphi_q - \varphi_{q-1} \partial'_q \colon C'_q \to \bar{C}_{q-1}$$

and

$$\begin{aligned} \tau_{q-1} h_q &= \tau_{q-1} \bar{\partial}_q \varphi_q - \tau_{q-1} \varphi_{q-1} \partial'_q = \partial_q \tau_q \varphi_q - \tau'_{q-1} \partial'_q \\ &= \partial_q \tau'_q - \tau'_{q-1} \partial'_q = 0 \end{aligned}$$

Therefore h_q is a homomorphism of C'_q into $i(\bar{\bar{C}}_{q-1})$. It follows immediately that $\bar{\tau} = \{\bar{\tau}_q = \varphi_q - iD_{q-1} i^{-1} h_q\}$ is a chain map $\bar{\tau}$: $C' \to \bar{C}$ such that $\tau \bar{\tau} = \tau'$.

If $\bar{\tau}$, $\bar{\tau}'$: $C' \to \bar{C}$ are chain maps such that $\tau \bar{\tau} = \tau \bar{\tau}'$, then $\bar{\tau} - \bar{\tau}' = i\psi$ for some chain map ψ: $C' \to \bar{\bar{C}}$. It follows immediately that

$$\bar{D} = \{\bar{D}_q = iD_q \psi_q \colon C'_q \to \bar{C}_{q+1}\}$$

is a chain homotopy from $\bar{\tau}$ to $\bar{\tau}'$. ∎

If C is a chain complex over R and G is an R module, let $C * G$ be the chain complex $C * G = \{C_q * G, \partial_q * 1\}$. We use this in the general universal-coefficient theorem.

14 THEOREM *On the subcategory of the product category of chain complexes C and modules G such that $C * G$ is acyclic there is a functorial short exact sequence*

$$0 \to H_q(C) \otimes G \xrightarrow{\mu} H_q(C;G) \to H_{q-1}(C) * G \to 0$$

and this sequence is split.

PROOF Let τ: $\bar{C} \to C$ be a free approximation to C (which exists, by lemma 12), and consider the short exact sequence

$$0 \to \bar{\bar{C}} \xrightarrow{i} \bar{C} \xrightarrow{\tau} C \to 0$$

in which $\bar{\bar{C}}$ is acyclic. By the characteristic property of the torsion product, there is an exact sequence of chain complexes

$$0 \to C * G \to \bar{\bar{C}} \otimes G \xrightarrow{i \otimes 1} \bar{C} \otimes G \xrightarrow{\tau \otimes 1} C \otimes G \to 0$$

from which we get two short exact sequences

$$0 \to C * G \to \bar{\bar{C}} \otimes G \to \operatorname{im}(i \otimes 1) \to 0$$

$$0 \to \operatorname{im}(i \otimes 1) \subset \bar{C} \otimes G \xrightarrow{\tau \otimes 1} C \otimes G \to 0$$

In the first of these $C * G$ is acyclic by hypothesis, and $\bar{\bar{C}} \otimes G$ is also acyclic (by theorem 8, because $\bar{\bar{C}}$ is free and acyclic). From corollary 4.5.5*c* it follows that $\operatorname{im}(i \otimes 1)$ is also acyclic. In the second exact homology sequence this implies that

$$(\tau \otimes 1)_*: H(\bar{C} \otimes G) \approx H(C \otimes G)$$

The desired short exact sequence is now defined, so that the following diagram is commutative

$$\begin{array}{ccccccccc}
0 & \to & H_q(\bar{C}) \otimes G & \xrightarrow{\mu} & H_q(\bar{C} \otimes G) & \to & H_{q-1}(\bar{C}) * G & \to & 0 \\
 & & \downarrow{\scriptstyle \tau_* \otimes 1} & & \downarrow{\scriptstyle (\tau \otimes 1)_*} & & \downarrow{\scriptstyle \tau_* * 1} & & \\
0 & \to & H_q(C) \otimes G & \xrightarrow{\mu} & H_q(C \otimes G) & \to & H_{q-1}(C) * G & \to & 0
\end{array}$$

where the upper row is the short exact sequence of theorem 8 (it is possible to define the unlabeled homomorphism in the bottom sequence to make the diagram commutative because all the vertical homomorphisms are isomorphisms). Then the bottom sequence splits because the top one does.

The functorial property of the resulting short exact sequence (and the fact that it is independent of the particular free approximation of C) follows from lemma 13. ∎

It should be emphasized again that the sequence of theorem 14 does not split functorially.

15 COROLLARY *Let $\tau: C \to C'$ be a chain map between torsion-free chain complexes such that $\tau_*: H(C) \approx H(C')$. For any R module G, τ induces an isomorphism*

$$\tau_*: H(C;G) \approx H(C';G)$$

PROOF This follows from the functorial exact sequence of theorem 14 and the five lemma. ∎

In corollary 15, if C and C' are free, then τ is a chain equivalence (by theorem 4.6.10), and so is $\tau \otimes 1: C \otimes G \to C' \otimes G$. Therefore $\tau_*: H(C;G) \approx H(C';G)$. Corollary 15 shows that the latter fact remains true (even though τ need not be a chain equivalence) for chain complexes without torsion.

3 THE KÜNNETH FORMULA

In this section we extend the universal-coefficient theorem to obtain the Künneth formula expressing the homology of the tensor product of two chain complexes in terms of the homology of the factors. This is given geometric content by the Eilenberg-Zilber theorem asserting that the singular complex of a product space is chain equivalent to the tensor product of the singular complexes of the factor spaces.

If C and C' are graded R modules, their *tensor product* $C \otimes C'$ is the graded module $\{(C \otimes C')_q\}$, where $(C \otimes C')_q = \bigoplus_{i+j=q} C_i \otimes C'_j$. Similarly, their *torsion product* $C * C'$ is the graded module $\{(C * C')_q = \bigoplus_{i+j=q} C_i * C'_j\}$. If C and C' are chain complexes, their tensor product [and torsion product] are chain complexes $\{(C \otimes C')_q, \partial''_q\}$ [and $\{(C * C')_q, \bar\partial_q\}$], where if $c \in C_i$ and $c' \in C_j$ with $i + j = q$, then

$$\partial''_q(c \otimes c') = \partial_i c \otimes c' + (-1)^i c \otimes \partial'_j c'$$

[and $\bar\partial_q \mid C_i * C'_j = \partial_i * 1 + (-1)^i 1 * \partial'_j$]. It is easy to verify that $C \otimes C'$ [and $C * C'$] really are chain complexes. We shall see later that the tensor product arises naturally in studying product spaces.

If C' is a chain complex such that $C'_q = 0$ for $q \neq 0$, then $C \otimes C'$ is the same as the tensor product of C with the module C'_0. Therefore the tensor product of two chain complexes is a natural generalization of the tensor product of a chain complex with a module. It is reasonable to expect that there is a generalization of the universal-coefficient theorem to express the homology of $C \otimes C'$ in terms of the homology of C and of C'.

We define a functorial homomorphism of degree 0

$$\mu\colon H(C) \otimes H(C') \to H(C \otimes C')$$

If $c \in Z_i(C)$ and $c' \in Z_j(C')$, then $c \otimes c' \in Z_{i+j}(C \otimes C')$, and if c or c' is a boundary, so is $c \otimes c'$. Therefore there is a well-defined homomorphism μ such that

$$\mu(\{c\} \otimes \{c'\}) = \{c \otimes c'\}$$

This homomorphism enters in the following *Künneth formula*.

1 LEMMA *Let C and C' be chain complexes, with C' free. Then there is a functorial short exact sequence*

$$0 \to [H(C) \otimes H(C')]_q \xrightarrow{\mu} H_q(C \otimes C') \to [H(C) * H(C')]_{q-1} \to 0$$

If C is also free, this short exact sequence is split.

PROOF As in the proof of theorem 5.2.8, let Z' and B' be the complexes (with trivial boundary operators) defined by $Z'_q = Z_q(C')$ and $B'_q = B_{q-1}(C')$. There is a short exact sequence of chain complexes

$$0 \to Z' \to C' \to B' \to 0$$

Since C' is free, so is B', and there is a short exact sequence

$$0 \to C \otimes Z' \to C \otimes C' \to C \otimes B' \to 0$$

from which we obtain an exact homology sequence

$$\cdots \to H_q(C \otimes Z') \to H_q(C \otimes C') \to H_q(C \otimes B') \xrightarrow{\partial_*} H_{q-1}(C \otimes Z') \to \cdots$$

Note that $C \otimes Z' = \bigoplus C^j$, where $(C^j)_q = C_{q-j} \otimes Z_j(C')$ and $C \otimes B' = \bigoplus \bar{C}^j$, where $(\bar{C}^j)_q = C_{q-j} \otimes B_{j-1}(C')$. Since $Z_j(C')$ and $B_j(C')$ are free, it follows from theorem 5.2.14 that

$$H_q(C \otimes Z') = \bigoplus_j H_q(C^j) = \bigoplus_{i+j=q} H_i(C) \otimes Z_j(C')$$

$$H_q(C \otimes B') = \bigoplus_j H_q(\bar{C}^j) = \bigoplus_{i+j=q-1} H_i(C) \otimes B_j(C')$$

The map ∂_* corresponds under these isomorphisms to the homomorphism $(-1)^i \otimes \gamma_j$, where γ_j is the inclusion map γ_j: $B_j(C') \subset Z_j(C')$. Therefore there is a short exact sequence

$$0 \to \bigoplus_{i+j=q} [\operatorname{coker}(-1)^i \otimes \gamma_j] \to H_q(C \otimes C') \to \bigoplus_{i+j=q-1} [\ker(-1)^i \otimes \gamma_j] \to 0$$

To compute the two sides of this sequence, consider the short exact sequence

$$0 \to B_j(C') \xrightarrow{(-1)^i\gamma_j} Z_j(C') \to H_j(C') \to 0$$

Because $Z_j(C')$ is free, it follows from corollary 5.2.9 that there is an exact sequence

$$0 \to H_i(C) * H_j(C') \to H_i(C) \otimes B_j(C') \xrightarrow{(-1)^i \otimes \gamma_j} H_i(C) \otimes Z_j(C') \to H_i(C) \otimes H_j(C') \to 0$$

Hence

$$\bigoplus_{i+j=q} [\operatorname{coker}(-1)^i \otimes \gamma_j] = \bigoplus_{i+j=q} H_i(C) \otimes H_j(C')$$

and

$$\bigoplus_{i+j=q-1} [\ker(-1)^i \otimes \gamma_j] = \bigoplus_{i+j=q-1} H_i(C) * H_j(C')$$

Substituting these into the short exact sequence above gives a short exact sequence

$$0 \to [H(C) \otimes H(C')]_q \xrightarrow{\nu} H_q(C \otimes C') \to [H(C) * H(C')]_{q-1} \to 0$$

We now verify that ν is the map μ. Given $\{c\} \in H(C)$ and $\{c'\} \in H(C')$, then $\{c\} \otimes c' \in H(C) \otimes Z(C')$ and $\{c\} \otimes c' = \{c \otimes c'\}_{C \otimes Z(C')}$. Therefore $\nu(\{c\} \otimes \{c'\}) = \{c \otimes c'\}_{C \otimes C'} = \mu(\{c\} \otimes \{c'\})$. Thus we have the desired short exact sequence, and it is clearly functorial.

Assuming that C is also free, we can show that the sequence splits. By lemma 5.1.11, it suffices to find a left inverse for μ. Because C and C' are free, so are $B(C)$ and $B(C')$, and there are homomorphisms $p: C \to Z(C)$ and p': $C' \to Z(C')$ such that $p(c) = c$ for $c \in Z(C)$ and $p'(c') = c'$ for $c' \in Z(C')$. Then

$$p \otimes p': C \otimes C' \to Z(C) \otimes Z(C')$$

maps $B(C \otimes C')$(which is contained in the union of im $[B(C) \otimes C' \to C \otimes C']$ and im $[C \otimes B(C') \to C \otimes C']$) into the union of im $[B(C) \otimes Z(C') \to Z(C) \otimes Z(C')]$ and im $[Z(C) \otimes B(C') \to Z(C) \otimes Z(C')]$. Therefore the composite

$$Z(C \otimes C') \subset C \otimes C' \xrightarrow{p \otimes p'} Z(C) \otimes Z(C') \to H(C) \otimes H(C')$$

maps $B(C \otimes C')$ into 0 and induces a homomorphism

$$H(C \otimes C') \to H(C) \otimes H(C')$$

which is a left inverse of μ. ∎

A similar functorial short exact sequence can be defined if C (instead of C') is assumed free. The two short exact sequences are identical when C and C' are both free.[1]

2 COROLLARY *If C' is a free chain complex and either C or C' is acyclic, then $C \otimes C'$ is acyclic.* ∎

We now extend lemma 1 to obtain the following general Künneth formula.

3 THEOREM *On the subcategory of the product category of chain complexes C and C' such that $C * C'$ is acyclic there is a functorial short exact sequence*

$$0 \to [H(C) \otimes H(C')]_q \xrightarrow{\mu} H_q(C \otimes C') \to [H(C) * H(C')]_{q-1} \to 0$$

and this sequence is split.

PROOF Let $\tau\colon \bar{C} \to C$ and $\tau'\colon \bar{C}' \to C'$ be free approximations. Then there is a short exact sequence

$$0 \to \bar{\bar{C}}' \xrightarrow{i'} \bar{C}' \xrightarrow{\tau'} C' \to 0$$

where $\bar{\bar{C}}'$ is acyclic. Since $\bar{C}'$ is free, the six-term exact sequence becomes the exact sequence

$$0 \to C * C' \to C \otimes \bar{\bar{C}}' \to C \otimes \bar{C}' \xrightarrow{1 \otimes \tau'} C \otimes C' \to 0$$

Since $C * C'$ is acyclic by hypothesis and $C \otimes \bar{\bar{C}}'$ is acyclic by corollary 2, it follows (as in the proof of theorem 5.2.14) that there is an isomorphism

$$(1 \otimes \tau')_*\colon H(C \otimes \bar{C}') \approx H(C \otimes C')$$

There is also a short exact sequence

$$0 \to \bar{\bar{C}} \xrightarrow{i} \bar{C} \xrightarrow{\tau} C \to 0$$

where $\bar{\bar{C}}$ is acyclic. Since $\bar{C}'$ is free, there is a short exact sequence

$$0 \to \bar{\bar{C}} \otimes \bar{C}' \to \bar{C} \otimes \bar{C}' \xrightarrow{\tau \otimes 1} C \otimes \bar{C}' \to 0$$

By corollary 2, $\bar{\bar{C}} \otimes \bar{C}'$ is acyclic, and we have an isomorphism

$$(\tau \otimes 1)_*\colon H(\bar{C} \otimes \bar{C}') \approx H(C \otimes \bar{C}')$$

[1] This is proved in G. M. Kelley, Observations on the Künneth theorem, *Proceedings of the Cambridge Philosophical Society*, vol. 59, pp. 575–587, 1963.

Hence the composite $(\tau \otimes \tau')_* = (1 \otimes \tau')_*(\tau \otimes 1)_*$ is an isomorphism of $H(\bar{C} \otimes \bar{C}')$ onto $H(C \otimes C')$. The desired short exact sequence is now defined so that the following diagram is commutative

$$\begin{array}{ccccccccc} 0 & \to & H(\bar{C}) \otimes H(\bar{C}') & \xrightarrow{\mu} & H(\bar{C} \otimes \bar{C}') & \to & H(\bar{C}) * H(\bar{C}') & \to & 0 \\ & & \downarrow {\scriptstyle \tau_* \otimes \tau'_*} & & \downarrow {\scriptstyle (\tau \otimes \tau')_*} & & \downarrow {\scriptstyle \tau_* * \tau'_*} & & \\ 0 & \to & H(C) \otimes H(C') & \xrightarrow{\mu} & H(C \otimes C') & \to & H(C) * H(C') & \to & 0 \end{array}$$

where the top row is the short exact sequence of lemma 1 (it is possible to define the homomorphisms in the bottom row to make the diagram commutative because the vertical homomorphisms are isomorphisms). The bottom sequence splits because the top one does.

The functorial property of the sequence (and the fact that it is independent of the free approximations $\bar{C}$ and $\bar{C}'$) follow from the functorial property of the sequence in lemma 1 and from lemma 5.2.13. ∎

If C and C' are chain complexes over R and G and G' are R modules, the composite

$$H(C \otimes G) \otimes H(C' \otimes G') \xrightarrow{\mu} H[(C \otimes G) \otimes (C' \otimes G')] \to H[(C \otimes C') \otimes (G \otimes G')]$$

[where the right-hand homomorphism is induced by the canonical isomorphism $(C \otimes G) \otimes (C' \otimes G') \approx (C \otimes C') \otimes (G \otimes G')$] is a functorial homomorphism

$$\mu': H(C;G) \otimes H(C';G') \to H(C \otimes C'; G \otimes G')$$

called the *cross product.* If $z \in H(C;G)$ and $z' \in H(C';G')$, then

$$z \times z' \in H(C \otimes C'; G \otimes G')$$

denotes the image of $z \otimes z'$ under this homomorphism [that is, $z \times z' = \mu'(z \otimes z')$].

4 COROLLARY *Given torsion-free chain complexes C and C' and modules G and G' such that $G * G' = 0$, there is a functorial short exact sequence*

$$0 \to [H(C;G) \otimes H(C';G')]_q \xrightarrow{\mu'} H_q(C \otimes C'; G \otimes G') \to [H(C;G) * H(C',G')]_{q-1} \to 0$$

and this sequence is split.

PROOF This follows from theorem 3 once we verify that $(C \otimes G) * (C' \otimes G')$ is trivial. To show that $(C \otimes G) * (C' \otimes G') = 0$, let $0 \to F' \to F \to G$ be a free presentation of G. Because $G * G' = 0$, there is an exact sequence

$$0 \to F' \otimes G' \to F \otimes G' \to G \otimes G' \to 0$$

and since C and C' are without torsion, there is an exact sequence

$$0 \to (C \otimes F') \otimes (C' \otimes G') \to (C \otimes F) \otimes (C' \otimes G') \to (C \otimes G) \otimes (C' \otimes G') \to 0$$

Because there is also a short exact sequence

$$0 \to C \otimes F' \to C \otimes F \to C \otimes G \to 0$$

where $C \otimes F$ is without torsion, it follows that $(C \otimes G) * (C' \otimes G')$ is isomorphic to the kernel of the homomorphism

$$(C \otimes F') \otimes (C' \otimes G') \to (C \otimes F) \otimes (C' \otimes G')$$

and hence is 0. ∎

The cross product has the following commutativity with connecting homomorphisms.

5 THEOREM *Let $0 \to \bar{\bar{C}} \to \bar{C} \to C \to 0$ be a split short exact sequence of chain complexes and let $z \in H(C;G)$ and $z' \in H(C';G')$. Then*

$$\partial_*(z \times z') = \partial_* z \times z'$$
$$\partial_*(z' \times z) = (-1)^{\deg z'} z' \times \partial_* z$$

PROOF We have a commutative diagram of chain maps

$$\begin{array}{ccccccccc}
0 \to & \bar{\bar{C}} \otimes G & \to & \bar{C} \otimes G & \to & C \otimes G & \to 0 \\
& {\scriptstyle \bar{\bar{\tau}}} \downarrow & & \downarrow {\scriptstyle \bar{\tau}} & & \downarrow {\scriptstyle \tau} & \\
0 \to & (\bar{\bar{C}} \otimes G) \otimes (C' \otimes G') & \to & (\bar{C} \otimes G) \otimes (C' \otimes G') & \to & (C \otimes G) \otimes (C' \otimes G') & \to 0
\end{array}$$

with exact rows, with the vertical maps defined by forming the tensor product on the right with $c' \in Z(C' \otimes G')$, where $z' = \{c'\}$ [that is, $\tau(c) = c \otimes c'$ for $c \in C \otimes G$]. Because c' is a cycle, each vertical map is a chain map. Because the connecting homomorphism is functorial, we obtain a commutative diagram

$$\begin{array}{ccccc}
H(C \otimes G) & \xrightarrow{\tau_*} & H((C \otimes G) \otimes (C' \otimes G')) & \underset{\approx}{\to} & H((C \otimes C') \otimes (G \otimes G')) \\
{\scriptstyle \partial_*} \downarrow & & {\scriptstyle \partial_*} \downarrow & & \downarrow {\scriptstyle \partial_*} \\
H(\bar{\bar{C}} \otimes G) & \xrightarrow{\bar{\bar{\tau}}_*} & H((\bar{\bar{C}} \otimes G) \otimes (C' \otimes G')) & \underset{\approx}{\to} & H((\bar{\bar{C}} \otimes C') \otimes (G \otimes G'))
\end{array}$$

in which each vertical map is a suitable connecting homomorphism. The top row sends z into $z \times z'$, and the bottom row sends $\partial_* z$ into $\partial_* z \times z'$. This gives half the result. The second half follows by a similar argument, the only difference being that the tensor product formed on the left with c' is not a chain map but either commutes or anticommutes with the boundary operator, depending on the degree of c'. This accounts for the presence of the factor $(-1)^{\deg z'}$ in the second equation. ∎

The following *Eilenberg-Zilber theorem*[1] is the link between the algebra of tensor products and the geometry of product spaces.

6 THEOREM *On the category of ordered pairs of topological spaces X and Y there is a natural chain equivalence of the functor $\Delta(X \times Y)$ with the functor $\Delta(X) \otimes \Delta(Y)$.*

[1] The theorem appears in S. Eilenberg and J. A. Zilber, On products of complexes, *American Journal of Mathematics*, vol. 75, pp. 200–204, 1953.

PROOF We show that both functors are free and acyclic with models $\{\Delta^p,\Delta^q\}_{p,q\geq 0}$. Let $d_n \in \Delta_n(\Delta^n \times \Delta^n)$ be the singular simplex which is the diagonal map $\Delta^n \to \Delta^n \times \Delta^n$. If $\sigma: \Delta^n \to X \times Y$ is any singular n-simplex, then σ is the composite

$$\Delta^n \xrightarrow{d_n} \Delta^n \times \Delta^n \xrightarrow{\sigma' \times \sigma''} X \times Y$$

where $\sigma' = p_1 \circ \sigma$ and $\sigma'' = p_2 \circ \sigma$, and p_1 and p_2 are the projections of $X \times Y$ to X and Y, respectively. Conversely, given $\sigma': \Delta^n \to X$ and $\sigma'': \Delta^n \to Y$, there is a corresponding $\sigma = (\sigma' \times \sigma'')d_n: \Delta^n \to X \times Y$. Therefore the singleton $\{d_n\}$ is a basis for $\Delta_n(X \times Y)$, so $\Delta(X \times Y)$ is free with models $\{\Delta^n,\Delta^n\}$, and hence also free with models $\{\Delta^p,\Delta^q\}$. Since Δ^p and Δ^q are each contractible, so is $\Delta^p \times \Delta^q$. Therefore $\tilde{\Delta}(\Delta^p \times \Delta^q)$ is acyclic, and we have proved that $\Delta(X \times Y)$ is a free acyclic functor with models $\{\Delta^p,\Delta^q\}$.

Since $\Delta_p(X)$ is free with a basis $\xi_p \in \Delta_p(\Delta^p)$ and $\Delta_q(Y)$ is free with basis $\xi_q \in \Delta_q(\Delta^q)$, it follows that $\Delta_p(X) \otimes \Delta_q(Y)$ is free with the basis

$$\xi_p \otimes \xi_q \in \Delta_p(\Delta^p) \otimes \Delta_q(\Delta^q).$$

Then $[\Delta(X) \otimes \Delta(Y)]_n$ is free with the basis $\{\xi_p \otimes \xi_q\}_{p+q=n}$. Hence $\Delta(X) \otimes \Delta(Y)$ is free with models $\{\Delta^p,\Delta^q\}$. Since $\varepsilon: \Delta(\Delta^p) \to \mathbf{Z}$ and $\varepsilon: \Delta(\Delta^q) \to \mathbf{Z}$ are chain equivalences, it follows that

$$\varepsilon \otimes \varepsilon: \Delta(\Delta^p) \otimes \Delta(\Delta^q) \to \mathbf{Z} \otimes \mathbf{Z} = \mathbf{Z}$$

is also a chain equivalence. Hence, by lemma 4.3.2, the reduced complex of $\Delta(\Delta^p) \otimes \Delta(\Delta^q)$ is acyclic, and we have shown that $\Delta(X) \otimes \Delta(Y)$ is also free and acyclic with models $\{\Delta^p,\Delta^q\}$. The theorem now follows by the method of acyclic models. ∎

The same technique based on the method of acyclic models can be used to prove the following results.

7 THEOREM *Given X, Y, and Z, there is a chain homotopy commutative diagram*

$$\begin{array}{ccc} \Delta((X \times Y) \times Z) & \approx & \Delta(X \times (Y \times Z)) \\ \approx\updownarrow & & \updownarrow\approx \\ [\Delta(X) \otimes \Delta(Y)] \otimes \Delta(Z) & \approx & \Delta(X) \otimes [\Delta(Y) \otimes \Delta(Z)] \end{array}$$

where the vertical maps are the natural chain equivalences of theorem 6. ∎

8 THEOREM *For any X and Y there is a chain homotopy commutative diagram*

$$\begin{array}{ccc} \Delta(X \times Y) & \approx & \Delta(Y \times X) \\ \approx\updownarrow & & \updownarrow\approx \\ \Delta(X) \otimes \Delta(Y) & \approx & \Delta(Y) \otimes \Delta(X) \end{array}$$

where the bottom map sends $x \otimes y$ *to* $(-1)^{\deg x \deg y} y \otimes x$ *and the vertical maps are the natural chain equivalences of theorem 6.* ∎

The sign in theorem 8 is necessary to make the map a chain map (that is, to make it commute with the boundary operators).

Given topological pairs (X,A) and (Y,B), we define their *product* $(X,A) \times (Y,B)$ to be the pair $(X \times Y, X \times B \cup A \times Y)$. Then we have the following relative form of the Eilenberg-Zilber theorem.

9 THEOREM *On the category of ordered pairs of topological pairs* (X,A) *and* (Y,B) *such that* $\{X \times B, A \times Y\}$ *is an excisive couple in* $X \times Y$ *there is a natural chain equivalence of* $\Delta(X \times Y)/\Delta(X \times B \cup A \times Y)$ *with* $[\Delta(X)/\Delta(A)] \otimes [\Delta(Y)/\Delta(B)]$.

PROOF Because $\{X \times B, A \times Y\}$ is an excisive couple, the natural map

$$\Delta(X \times Y)/[\Delta(X \times B) + \Delta(A \times Y)] \to \Delta(X \times Y)/\Delta(X \times B \cup A \times Y)$$

is a chain equivalence. By theorem 6 there is a functorial equivalence of $\Delta(X) \otimes \Delta(Y)$ with $\Delta(X \times Y)$ taking $\Delta(X) \otimes \Delta(B)$ and $\Delta(A) \otimes \Delta(Y)$ into $\Delta(X \times B)$ and $\Delta(A \times Y)$, respectively. Hence there is a functorial chain equivalence of the quotient

$$\Delta(X) \otimes \Delta(Y)/[\Delta(X) \otimes \Delta(B) + \Delta(A) \otimes \Delta(Y)] \approx [\Delta(X)/\Delta(A)] \otimes [\Delta(Y)/\Delta(B)]$$

with the quotient

$$\Delta(X \times Y)/[\Delta(X \times B) + \Delta(A \times Y)]$$

Combining these two chain equivalences gives the result. ∎

For any two pairs (X,A) and (Y,B) we define the *homology cross product*

$$\mu': H_p(X,A;\, G) \otimes H_q(Y,B;\, G') \to H_{p+q}((X,A) \times (Y,B);\, G \otimes G')$$

to be equal to the cross product

$$\begin{gathered} H_p([\Delta(X)/\Delta(A)] \otimes G) \otimes H_q([\Delta(Y)/\Delta(B)] \otimes G') \\ \downarrow \\ H_{p+q}(([\Delta(X)/\Delta(A)] \otimes [\Delta(Y)/\Delta(B)]) \otimes (G \otimes G')) \end{gathered}$$

followed by the functorial homomorphism of the bottom module to

$$H_{p+q}(\Delta(X \times Y)/\Delta(X \times B \cup A \times Y);\, G \otimes G')$$

If $z \in H_p(X,A;\, G)$ and $z' \in H_q(Y,B;\, G')$, then we write

$$z \times z' = \mu'(z \otimes z') \in H_{p+q}((X,A) \times (Y,B);\, G \otimes G')$$

Because $\Delta(X)/\Delta(A)$ and $\Delta(Y)/\Delta(B)$ are free, we can combine theorem 9 with corollary 4 to obtain the following Künneth formula for singular homology.

10 THEOREM *If $\{X \times B, A \times Y\}$ is an excisive couple in $X \times Y$ and G and G' are modules over a principal ideal domain such that $G * G' = 0$, there is a functorial short exact sequence*

$$0 \to [H(X,A;\, G) \otimes H(Y,B;\, G')]_q \xrightarrow{\mu'} H_q((X,A) \times (Y,B);\, G \otimes G') \to [H(X,A;\, G) * H(Y,B;\, G')]_{q-1} \to 0$$

and this sequence is split. ■

In particular, if the right-hand term vanishes (which always happens if R is a field), then the cross product is an isomorphism

$$\mu' \colon H(X,A;\, G) \otimes H(Y,B;\, G') \approx H((X,A) \times (Y,B);\, G \otimes G')$$

The following formulas are consequences of the naturality of μ and of theorems 5, 7, and 8.

11 *Let $f\colon (X,A) \to (X',A')$ and $g\colon (Y,B) \to (Y',B')$ be maps and let $z \in H_p(X,A;\, G)$ and $z' \in H_q(Y,B;\, G')$. Then, in the module*

$$H_{p+q}((X',A') \times (Y',B');\, G \otimes G')$$

we have

$$(f \times g)_*(z \times z') = f_* z \times g_* z' \quad ■$$

12 *Let $p\colon (X,A) \times Y \to (X,A)$ be the projection to the first factor and let $\varepsilon\colon H(Y;G') \to G'$ be the augmentation map. For $z \in H_q(X,A;\, G)$ and $z' \in H_r(Y;G')$, in $H_{q+r}(X,A;\, G \otimes G')$,*

$$p_*(z \times z') = \mu(z \otimes \varepsilon(z')) \quad ■$$

13 *For $z \in H_p(X,A;\, G)$, $z' \in H_q(Y,B;\, G')$, and $z'' \in H_r(Z,C;\, G'')$, in*

$$H_{p+q+r}((X,A) \times (Y,B) \times (Z,C);\, G \otimes G' \otimes G''),$$

we have

$$z \times (z' \times z'') = (z \times z') \times z'' \quad ■$$

14 *Let $T\colon (X,A) \times (Y,B) \to (Y,B) \times (X,A)$ and $\varphi\colon G' \otimes G \to G \otimes G'$ interchange the factors. For $z \in H_p(X,A;\, G)$ and $z' \in H_q(Y,B;\, G')$, in $H_{p+q}((Y,B) \times (X,A);\, G \otimes G')$, we have*

$$T_*(z \times z') = (-1)^{pq} \varphi_*(z' \times z) \quad ■$$

15 *Let $\{(X_1,A_1), (X_2,A_2)\}$ be an excisive couple of pairs in X and let $z \in H_p(X_1 \cup X_2, A_1 \cup A_2;\, G)$ and $z' \in H_q(Y,B;\, G')$. For the connecting homomorphisms of appropriate Mayer-Vietoris sequences we have*

$$\partial_*(z \times z') = \partial_* z \times z'$$

in $H_{p+q-1}((X_1 \cap X_2, A_1 \cap A_2) \times (Y,B);\, G \otimes G')$ and

$$\partial_*(z' \times z) = (-1)^q z' \times \partial_* z$$

in $H_{p+q-1}((Y,B) \times (X_1 \cap X_2, A_1 \cap A_2);\, G' \otimes G)$ ■

4 COHOMOLOGY

A chain complex has a differential of degree -1. Related to this is the concept of a cochain complex, which has a differential of degree $+1$. Cochain complexes have many of the properties of chain complexes, and this section is devoted to a discussion of these properties. The functor Hom associates to every chain complex a cochain complex, and vice versa. The cohomology module of a topological pair with coefficients G is the homology module of the cochain complex associated in this way to the singular complex of the pair. The last part of the section is devoted to a brief discussion of axiomatic cohomology theory.

Throughout this section R will denote a commutative ring with a unit. A *cochain complex* (*over* R), denoted by $C^* = \{C^q,\delta^q\}$, is a graded R module together with a homogeneous differential δ of degree $+1$ called the *coboundary operator* (thus $\delta^q\colon C^q \to C^{q+1}$ and $\delta^{q+1}\delta^q = 0$ for all q). The kernel of δ is the module of *cocycles* $Z(C^*)$, and the image of δ is the module of *coboundaries* $B(C^*)$. Then $B(C^*) \subset Z(C^*)$, and the *cohomology module* $H(C^*)$ is defined to be the quotient $Z(C^*)/B(C^*)$.

If C^* is a cochain complex, there is a chain complex C defined by $C_q = C^{-q}$ and $\partial_q\colon C_q \to C_{q-1}$ equal to $\delta^{-q}\colon C^{-q} \to C^{-q+1}$. Then $H_q(C) = H^{-q}(C^*)$, and the cohomology module of C^* corresponds to the homology module of C. In this way results about chain complexes give results about cochain complexes. Thus we have the concepts of *cochain map* and *cochain homotopy*, and there is a category of cochain complexes and cochain homotopy classes of cochain maps. The cohomology module is a covariant functor from this category to the category of graded modules. Furthermore, given a short exact sequence of cochain complexes

$$0 \to \bar{\bar{C}}^* \xrightarrow{\alpha} \bar{C}^* \xrightarrow{\beta} C^* \to 0$$

there is a functorial *connecting homomorphism*

$$\delta^*\colon H(C^*) \to H(\bar{\bar{C}}^*)$$

of degree $+1$ and a functorial exact cohomology sequence

$$\cdots \to H^q(C^*) \xrightarrow{\delta^*} H^{q+1}(\bar{\bar{C}}^*) \xrightarrow{\alpha^*} H^{q+1}(\bar{C}^*) \xrightarrow{\beta^*} H^{q+1}(C^*) \xrightarrow{\delta^*} \cdots$$

Passing from a cochain complex to a chain complex by changing the sign of the degree gives us the following analogues of theorems 5.2.14 and 5.3.3.

1 THEOREM *Given a cochain complex C^* and a module G such that $C^* * G$ is acyclic, there is a functorial short exact sequence*

$$0 \to H^q(C^*) \otimes G \xrightarrow{\mu} H^q(C^* \otimes G) \to H^{q+1}(C^*) * G \to 0$$

and this sequence is split. ∎

2 THEOREM *Given cochain complexes C^* and C'^* such that $C^* * C'^*$ is an acyclic cochain complex, there is a functorial short exact sequence*

$$0 \to [H^*(C^*) \otimes H^*(C'^*)]^q \xrightarrow{\mu} H^q(C^* \otimes C'^*) \to [H^*(C^*) * H^*(C'^*)]^{q+1} \to 0$$

and this sequence is split. ∎

There is also an analogue of corollary 5.3.4 for cochain complexes which we shall not state as a separate theorem. If C^* is a cochain complex over R and G is an R module, an *augmentation of* C^* *over* G is a monomorphism $\eta\colon G \to C^0$ such that $\delta^0 \circ \eta = 0$. An *augmented* cochain complex over G consists of a nonnegative cochain complex C^* (that is, $C^q = 0$ for $q < 0$) and an augmentation of C^* over G. Such an augmentation can be regarded as a monomorphic chain map of the cochain complex (also denoted by G) whose only nontrivial cochain module is G in degree 0 to C^*. For this trivial cochain complex G it is clear that $H^q(G) = 0$ for $q \neq 0$ and $H^0(G) = G$. Therefore η induces a monomorphism $\eta^*\colon G \to H^0(C^*)$. The *reduced cochain complex* $\tilde{C}^*$ of an augmented cochain complex C^* is defined to be the quotient cochain complex $\tilde{C}^q = C^q$ for $q \neq 0$, $\tilde{C}^0 = \operatorname{coker} \eta$, and $\tilde{\delta}^q$ is suitably induced by δ^q. We define $\tilde{H}(C^*) = H(\tilde{C}^*)$. Because there is a short exact sequence of cochain complexes

$$0 \to G \xrightarrow{\eta} C^* \to \tilde{C}^* \to 0$$

we see that $H^q(C^*) \approx \tilde{H}^q(C^*)$ for $q \neq 0$, and there is a short exact sequence

$$0 \to G \to H^0(C^*) \to \tilde{H}^0(C^*) \to 0$$

Our interest in cochain complexes is in their relation to chain complexes. If C is a chain complex over R and G is an R module, there is a cochain complex $\operatorname{Hom}(C,G) = \{\operatorname{Hom}(C_q,G), \delta^q\}$, where, if $f \in \operatorname{Hom}(C_q,G)$, then $\delta^q f \in \operatorname{Hom}(C_{q+1},G)$ is defined by

$$(\delta^q f)(c) = f(\partial_{q+1} c) \qquad c \in C_{q+1}$$

We also write $\langle f,c\rangle$ instead of $f(c)$ and set $\langle f,c\rangle = 0$ if $\deg f \neq \deg c$. In this notation $\langle \delta^q f,c\rangle = \langle f,\partial_{q+1} c\rangle$.

If C is augmented by $\varepsilon\colon C_0 \to G'$, then $\operatorname{Hom}(C,G)$ is augmented by $\eta\colon \operatorname{Hom}(G',G) \to \operatorname{Hom}(C_0,G)$, where $\eta(f)(c) = f(\varepsilon(c))$ for $c \in C_0$ and $f \in \operatorname{Hom}(G',G)$. It is easy to verify the following result.

3 THEOREM *There is a functor of two arguments contravariant in chain complexes C and covariant in modules G which assigns to a chain complex C and module G the cochain complex* $\operatorname{Hom}(C,G)$. ∎

For a chain complex C and module G we define the *cohomology module* $H^*(C;G) = \{H^q(C;G)\}$ *of C with coefficients* G by

$$H^q(C;G) = H^q(\text{Hom}\,(C,G))$$

It follows from theorem 3 that $H^*(C;G)$ is a contravariant functor of C and a covariant functor of G to the category of graded modules. For a chain map $\tau\colon C \to C'$ we use $\tau^*\colon H^*(C';G) \to H^*(C;G)$ to denote the homomorphism induced by the cochain map Hom $(\tau,1)$: Hom $(C',G) \to$ Hom (C,G), and for a homomorphism $\varphi\colon G \to G'$ we use $\varphi_*\colon H^*(C;G) \to H^*(C,G')$ to denote the homomorphism induced by the cochain map Hom $(1,\varphi)$: Hom $(C,G) \to$ Hom (C,G'). To distinguish the homology of C from its cohomology, we shall sometimes denote $H(C;G)$ by $H_*(C;G)$.

4 EXAMPLE Given an abelian group G and a simplicial pair (K,L), the *oriented cohomology group of* (K,L) *with coefficients* G, denoted by $H^*(K,L;\,G)$, is defined to be the graded cohomology group of the cochain complex Hom $(C(K)/C(L),\,G)$ [which is augmented over Hom $(\mathbf{Z},G) \approx G$]. Then $H^*(K,L;\,G)$ is a contravariant functor of (K,L) and a covariant functor of G to the category of graded abelian groups. If G is also an R module, $H^*(K,L;\,G)$ is a graded R module.

5 EXAMPLE If (X,A) is a topological pair and G is an abelian group, the *singular cohomology group of* (X,A) *with coefficients* G, denoted by $H^*(X,A;\,G)$, is defined to be the graded cohomology group of the cochain complex Hom $(\Delta(X)/\Delta(A),\,G)$ (which is augmented over G). It is contravariant in (X,A) and covariant in G, and if G is an R module, $H^*(X,A;\,G)$ is a graded R module. If $(X',A') \subset (X,A)$ and $u \in H^*(X,A;\,G)$, we use $u \mid (X',A')$ to denote the element of $H^*(X',A';\,G)$ equal to i^*u, where $i\colon (X',A') \subset (X,A)$. We also use $1 \in H^0(X;R)$ to denote the image of $1 \in R$ under the augmentation $\eta\colon R \to H^0(X;R)$.

We recall some properties of the functor Hom. The following analogue of corollary 5.1.6 is easily established.

6 LEMMA *Given an exact sequence of R modules*

$$A' \to A \to A'' \to 0$$

and an R module B, there is an exact sequence

$$0 \to \text{Hom}\,(A'',B) \to \text{Hom}\,(A,B) \to \text{Hom}\,(A',B) \quad \blacksquare$$

If $A' \to A$ is a monomorphism [that is, if $0 \to A' \to A$ is also exact], it need not be true that Hom $(A,B) \to$ Hom (A',B) is an epimorphism, [that is, that Hom $(A,B) \to$ Hom $(A',B) \to 0$ is exact]. However, there is the following analogue of corollary 5.1.12 (which follows easily by using lemma 5.1.11).

7 LEMMA *Given a split short exact sequence of R modules*

$$0 \to A' \to A \to A'' \to 0$$

and an R module B, the sequence

$$0 \to \text{Hom}\,(A'',B) \to \text{Hom}\,(A,B) \to \text{Hom}\,(A',B) \to 0$$

is a split short exact sequence. ■

In case $0 \to C' \to C \to C'' \to 0$ is a split short exact sequence of chain complexes, it follows from lemma 7 that for any module G there is a short exact sequence of cochain complexes

$$0 \to \text{Hom}\,(C'',G) \to \text{Hom}\,(C,G) \to \text{Hom}\,(C',G) \to 0$$

This gives the following result.

8 THEOREM *Given a split short exact sequence of chain complexes*

$$0 \to C' \to C \to C'' \to 0$$

and a module G, there is a functorial exact cohomology sequence

$$\cdots \to H^q(C'';G) \to H^q(C;G) \to H^q(C';G) \xrightarrow{\delta^*} H^{q+1}(C'';G) \to \cdots \quad ■$$

This leads to an exact *Mayer-Vietoris cohomology sequence* analogous to the exact sequence of corollary 5.1.14.

9 COROLLARY *If $\{(X_1,A_1), (X_2,A_2)\}$ is an excisive couple of pairs in a topological space and G is an R module, there is a functorial exact cohomology sequence*

$$\cdots \xrightarrow{\delta^*} H^q(X_1 \cup X_2, A_1 \cup A_2; G) \xrightarrow{j^*} H^q(X_1, A_1; G) \oplus H^q(X_2,A_2; G) \xrightarrow{i^*}$$
$$H^q(X_1 \cap X_2, A_1 \cap A_2; G) \xrightarrow{\delta^*} \cdots$$

where $j^*(u) = (j_1^*(u), j_2^*(u))$ and $i^*(u_1 + u_2) = i_1^* u_1 - i_2^* u_2$ *and* i_1, i_2, j_1, *and* j_2 *are suitable inclusion maps.* ■

If $\{X_j\}$ is the set of path components of X, then $\Delta(X) = \bigoplus_j \Delta(X_j)$. Therefore $\text{Hom}\,(\Delta(X);G) = \times_j \text{Hom}\,(\Delta(X_j),G)$, and by theorem 4.1.6, we obtain the following result.

10 THEOREM *The singular cohomology module of a space is the direct product of the singular cohomology modules of its path components.* ■

Because the homology functor does not commute with inverse limits, it is not true that the singular cohomology of a space is isomorphic to the inverse limit of the singular cohomology of its compact subsets (that is, there is no general cohomology analogue of theorem 4.4.6).

There is an exact cohomology sequence corresponding to a short exact sequence of coefficient modules (analogous to theorem 5.2.7).

11 THEOREM *On the category of free chain complexes C over R and short exact sequences of R modules*

$$0 \to G' \xrightarrow{\varphi} G \xrightarrow{\psi} G'' \to 0$$

there is a functorial connecting homomorphism

$$\beta^*: H^*(C;G'') \to H^*(C;G')$$

of degree 1 *and a functorial exact sequence*

$$\cdots \to H^q(C;G') \xrightarrow{\varphi_*} H^q(C;G) \xrightarrow{\psi_*} H^q(C;G'') \xrightarrow{\beta^*} H^{q+1}(C;G') \to \cdots$$

PROOF Because C is free, there is a short exact sequence of cochain complexes

$$0 \to \operatorname{Hom}(C,G') \xrightarrow{\bar\varphi} \operatorname{Hom}(C,G) \xrightarrow{\bar\psi} \operatorname{Hom}(C,G'') \to 0$$

where $\bar\varphi$ and $\bar\psi$ are induced by φ and ψ. The result follows from this. ∎

The connecting homomorphism β^* in theorem 11 is called the *Bockstein cohomology homomorphism* corresponding to the coefficient sequence $0 \to G' \xrightarrow{\varphi} G \xrightarrow{\psi} G'' \to 0$.

Let G be an R module. A *cohomology theory with coefficients* G consists of a contravariant functor $H^* = \{H^q\}$ from the category of topological pairs to graded R modules and a natural transformation $\delta^*: H^*(A) \to H^*(X,A)$ of degree 1 such that the following axioms hold.

12 HOMOTOPY AXIOM *If $f_0, f_1: (X,A) \to (Y,B)$ are homotopic, then*

$$H^*(f_0) = H^*(f_1): H^*(Y,B) \to H^*(X,A)$$

13 EXACTNESS AXIOM *For any pair (X,A) with inclusion maps $i: A \subset X$ and $j: X \subset (X,A)$, there is an exact sequence*

$$\cdots \xrightarrow{\delta^*} H^q(X,A) \xrightarrow{H^q(j)} H^q(X) \xrightarrow{H^q(i)} H^q(A) \xrightarrow{\delta^*} H^{q+1}(X,A) \to \cdots$$

14 EXCISION AXIOM *For any pair (X,A) if U is an open subset of X such that $\bar U \subset \operatorname{int} A$, then the excision map $j: (X - U, A - U) \subset (X,A)$ induces an isomorphism*

$$H^*(j): H^*(X,A) \approx H^*(X - U, A - U)$$

15 DIMENSION AXIOM *On the category of one-point spaces there is a natural equivalence of the constant functor G with the functor H^*.*

Singular cohomology theory with coefficients G is an example of a cohomology theory with coefficients G (the exactness axiom following from the application of corollary 9 to a suitable couple). The uniqueness theorem is valid for cohomology theories (that is, a homomorphism from one cohomology theory to another which is an isomorphism for one-point spaces is an isomorphism for compact polyhedral pairs).

The reduced cohomology modules $\tilde H^*$ of a cohomology theory are defined as follows. If X is a nonempty space, let $c: X \to P$ be the unique map from X to some one-point space P. The reduced module $\tilde H^*(X)$ is defined to be the cokernel of the homomorphism

$$H^*(c): H^*(P) \to H^*(X)$$

Because c has a right inverse, $H^*(c)$ has a left inverse. Therefore

$$H^*(X) \approx \tilde{H}^*(X) \oplus H^*(P)$$

and the reduced modules have properties similar to those of the reduced singular cohomology modules.

5 THE UNIVERSAL-COEFFICIENT THEOREM FOR COHOMOLOGY

This section is devoted to relations between cohomology and homology of chain complexes. In order to express the cohomology of a chain complex in terms of its homology it is necessary to introduce certain functors of modules which are associated to the module of homomorphisms of one module to another in much the same way that the torsion products are associated to the tensor product. Over a principal ideal domain there is one associated functor, the module of extensions. We use this to formulate the universal-coefficient theorems and Künneth formulas established in the section.

Let C be a free resolution of the module A and let B be a module. There is a cochain complex $\text{Hom}(C,B) = \{[\text{Hom}(C,B)]^q = \text{Hom}(C_q,B), \delta^q\}$ whose cohomology module depends only on A and B, up to canonical isomorphism (and not on the choice of C). Let C be the canonical free resolution of A and define $\text{Ext}^q(A,B) = H^q(\text{Hom}(C,B))$. Then $\text{Ext}^q(A,B)$ is a functor of two arguments contravariant in A and covariant in B, and $\text{Ext}^0(A,B)$ is naturally equivalent to $\text{Hom}(A,B)$.

For the rest of this section we assume R is a principal ideal domain. Then, $\text{Ext}^q(A,B) = 0$ for $q > 1$, and the *module of extensions* $\text{Ext}(A,B)$ is defined to equal $\text{Ext}^1(A,B)$. It is characterized by the property that given any free presentation of A

$$0 \to C_1 \xrightarrow{\partial_1} C_0 \to A \to 0$$

there is an exact sequence

$$0 \to \text{Hom}(A,B) \to \text{Hom}(C_0,B) \xrightarrow{\text{Hom}(\partial_1,1)} \text{Hom}(C_1,B) \to \text{Ext}(A,B) \to 0$$

In fact, because $\text{Hom}(C_2,B) = 0$,

$$\begin{aligned}\text{Ext}(A,B) = H^1(C;B) &= \text{Hom}(C_1,B)/\text{im}\,[\text{Hom}(\partial_1,1)] \\ &= \text{coker}\,[\text{Hom}(\partial_1,1)]\end{aligned}$$

Clearly, $\text{Ext}(A,B)$ is contravariant in A and covariant in B. Its name derives from its connection with the extensions of B by A which we describe briefly after the following examples.

1 If A is free, it has the free presentation

$$0 \to 0 \to A \to A \to 0$$

from which it follows that $\text{Ext}(A,B) = 0$ for any B.

2 For $v \in R$, $v \neq 0$ there is a short exact sequence

$$0 \to R \xrightarrow{\alpha} R \to R/vR \to 0$$

where $\alpha(v') = vv'$ for $v' \in R$, which is a free presentation of R/vR. For any R module B, $\operatorname{Hom}(R,B) \approx B$ and the homomorphism $\operatorname{Hom}(\alpha,1)$: $\operatorname{Hom}(R,B) \to \operatorname{Hom}(R,B)$ corresponds to $\alpha^*: B \to B$, where $\alpha^*(b) = vb$. Hence there is an isomorphism coker $\operatorname{Hom}(\alpha,1) \approx B/vB$, and we have proved

$$\operatorname{Ext}(R/vR,B) \approx B/vB \approx (R/vR) \otimes B$$

Since Hom commutes with finite direct sums, it follows that for any finitely generated torsion module A there is an isomorphism (nonfunctorial)

$$\operatorname{Ext}(A,B) \approx A \otimes B$$

because such a module A is a finite direct sum of cyclic modules (by theorem 4.14 in the Introduction).

An *extension of B by A* is a short exact sequence

$$0 \to B \to E \to A \to 0$$

With a suitable definition of equivalence of extensions (by a commutative diagram), of the sum of two extensions, and of the product of an extension by an element of R, there is obtained a module whose elements are equivalence classes of extensions of B by A. This module is isomorphic to $\operatorname{Ext}(A,B)$. In fact, given an extension $0 \to B \to E \to A \to 0$ and a free presentation of A, $0 \to C_1 \to C_0 \to A \to 0$, there is, by theorem 5.2.1, a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & C_1 \arrow[r] \arrow[d, "\varphi_1"'] & C_0 \arrow[d, "\varphi_0"'] \arrow[dr] & & \\
& & & A \arrow[r] & 0 \\
0 \arrow[r] & B \arrow[r] & E \arrow[ur] & &
\end{tikzcd}$$

uniquely determined up to chain homotopy. Then $\varphi_1 \in \operatorname{Hom}(C_1,B)$ is unique up to im $[\operatorname{Hom}(C_0,B) \to \operatorname{Hom}(C_1,B)]$, and so determines an element of $\operatorname{Ext}(A,B)$. This function from extensions of B by A to $\operatorname{Ext}(A,B)$ induces an isomorphism of the module of equivalence classes of extensions with $\operatorname{Ext}(A,B)$.

Given a graded module $C = \{C_q\}$, there is a graded module $\operatorname{Ext}(C,B) = \{[\operatorname{Ext}(C,B)]^q = \operatorname{Ext}(C_q,B)\}$. If C is a chain complex, $\operatorname{Ext}(C,B)$ is a cochain complex with

$$\delta^q = \operatorname{Ext}(\partial_{q+1},1)\colon \operatorname{Ext}(C_q,B) \to \operatorname{Ext}(C_{q+1},B)$$

A homomorphism

$$h\colon H^q(C;G) \to \operatorname{Hom}(H_q(C;G'), G \otimes G')$$

natural in C and G is defined by

$$(h\{f\})\{\textstyle\sum c_i \otimes g_i'\} = \textstyle\sum f(c_i) \otimes g_i'$$

for $\{f\} \in H^q(C;G)$ and $\{\sum c_i \otimes g_i'\} \in H_q(C;G')$ [after verification that

$\sum f(c_i) \otimes g'_i$ is independent of the choice of f in its cohomology class and $\sum c_i \otimes g'_i$ in its homology class]. For $u \in H^p(C;G)$ and $z \in H_q(C;G')$ we define $\langle u,z \rangle \in G \otimes G'$ to be 0 if $p \neq q$ and to be $h(u)(z)$ if $p = q$. In this notation

$$\langle \{f\}, \{\sum c_i \otimes g'_i\} \rangle = \sum \langle f,c_i \rangle \otimes g'_i$$

The homomorphism h enters in the following *universal-coefficient theorem for cohomology.*

3 THEOREM *Given a chain complex C and module G such that* Ext (C,G) *is an acyclic cochain complex, there is a functorial short exact sequence*

$$0 \to \operatorname{Ext}(H_{q-1}(C),G) \to H^q(C,G) \xrightarrow{h} \operatorname{Hom}(H_q(C),G) \to 0$$

and this sequence is split.

PROOF We first consider the case in which C is a free chain complex. There is then a short exact sequence of chain complexes

$$0 \to Z \to C \to B \to 0$$

where $Z_q = Z_q(C)$ and $B_q = B_{q-1}(C)$. This sequence is split because B is free, and by theorem 5.4.8, there is an exact cohomology sequence

$$\cdots \to H^{q-1}(Z;G) \xrightarrow{\delta^*} H^q(B;G) \to H^q(C;G) \to H^q(Z;G) \xrightarrow{\delta^*} H^{q+1}(B;G) \to \cdots$$

Since Z and B have trivial boundary operators, $H^q(Z;G) = \operatorname{Hom}(Z_q(C),G)$ and $H^q(B;G) = \operatorname{Hom}(B_{q-1}(C),G)$. Furthermore, the homomorphism

$$\delta^*: H^q(Z;G) \to H^{q+1}(B;G)$$

equals $\operatorname{Hom}(\gamma_q,1)$: $\operatorname{Hom}(Z_q(C),G) \to \operatorname{Hom}(B_q(C),G)$, where γ_q: $B_q(C) \subset Z_q(C)$. Hence there is a functorial short exact sequence

$$0 \to \operatorname{coker}[\operatorname{Hom}(\gamma_{q-1},1)] \to H^q(C;G) \to \ker[\operatorname{Hom}(\gamma_q,1)] \to 0$$

To interpret the modules in the above sequence we have the short exact sequence

$$0 \to B_q(C) \xrightarrow{\gamma_q} Z_q(C) \to H_q(C) \to 0$$

which is a free presentation of $H_q(C)$. By the characteristic property of Ext, there is an exact sequence

$$0 \to \operatorname{Hom}(H_q(C),G) \to \operatorname{Hom}(Z_q(C),G) \xrightarrow{\operatorname{Hom}(\gamma_q,1)}$$
$$\operatorname{Hom}(B_q(C),G) \to \operatorname{Ext}(H_q(C),G) \to 0$$

Therefore, $\ker[\operatorname{Hom}(\gamma_q,1)] \approx \operatorname{Hom}(H_q(C),G)$ and $\operatorname{coker}[\operatorname{Hom}(\gamma_q,1)] \approx \operatorname{Ext}(H_q(C),G)$. Substituting these into the short exact sequence containing $H^q(C;G)$ yields the desired short exact sequence

$$0 \to \operatorname{Ext}(H_{q-1}(C),G) \to H^q(C;G) \to \operatorname{Hom}(H_q(C),G) \to 0$$

with the homomorphism $H^q(C;G) \to \operatorname{Hom}(H_q(C),G)$ easily verified to equal h.

This sequence is functorial and is split (because the sequence of chain complexes

$$0 \to Z \to C \to B \to 0$$

is split).

For arbitrary C such that Ext (C,G) is acyclic, the result follows by using a free approximation to C (as in the proof of theorem 5.2.14) to reduce it to the case of a free complex. ■

It follows from theorem 3 that if X is a path-connected topological space, then $H^0(X;R)$ is a free R module generated by 1 [or, in other words, the augmentation map is an isomorphism η: $R \approx H^0(X;R)$]. From theorems 3 and 5.4.10, it follows that for any X, $H^0(X;G)$ is isomorphic to the direct product of as many copies of G as path components of X.

4 COROLLARY *If (X,A) is a topological pair such that $H_q(X,A;R)$ is finitely generated for all q, then the free submodules of $H^q(X,A;R)$ and $H_q(X,A;R)$ are isomorphic and the torsion submodules of $H^q(X,A;R)$ and $H_{q-1}(X,A;R)$ are isomorphic.*

PROOF Let $H_q(X,A;R) = F_q \oplus T_q$, where F_q is free and T_q is the torsion module of H_q. Then

$$\text{Hom}\,(H_q(X,A;R),R) \approx \text{Hom}\,(F_q,R) \oplus \text{Hom}\,(T_q,R) \approx F_q$$

and by example 2,

$$\text{Ext}\,(H_q(X,A;R),R) \approx \text{Ext}\,(F_q,R) \oplus \text{Ext}\,(T_q,R) \approx T_q$$

The result follows from theorem 3. ■

For many purposes it would be more useful to have a formula expressing $H^*(C;G)$ in terms of $H^*(C;R)$. Such a formula can be proved in the case of C or G finitely generated. We begin by establishing some properties of finitely generated modules.

Let μ: Hom $(A,G) \otimes G' \to$ Hom $(A, G \otimes G')$ be the functorial homomorphism defined by $\mu(f \otimes g')(a) = f(a) \otimes g'$ for $f \in$ Hom (A,G), $g' \in G'$, and $a \in A$.

5 LEMMA *If A is a free module and G' is finitely generated, then for any module G, μ is an isomorphism.*

PROOF The result is trivially true if $G' = R$. Because the tensor product and Hom functors both commute with finite direct sums, it is also true if G' is a finitely generated free module. G' is assumed to be finitely generated, so there is a short exact sequence

$$0 \to \bar{\bar{G}} \to \bar{G} \to G' \to 0$$

where $\bar{G}$ (hence also $\bar{\bar{G}}$) is a finitely generated free module. There is a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}\,(A,G) \otimes \bar{\bar{G}} \arrow[r] \arrow[d, "\bar{\bar{\mu}}"'] & \mathrm{Hom}\,(A,G) \otimes \bar{G} \arrow[r] \arrow[d, "\bar{\mu}"'] & \mathrm{Hom}\,(A,G) \otimes G' \arrow[r] \arrow[d, "\mu"'] & 0 \\
\mathrm{Hom}\,(A, G \otimes \bar{\bar{G}}) \arrow[r] & \mathrm{Hom}\,(A, G \otimes \bar{G}) \arrow[r] & \mathrm{Hom}\,(A, G \otimes G') \arrow[r] & 0
\end{tikzcd}$$

with exact rows (exactness follows from corollary 5.1.6 and, for the bottom row, from the fact that A is free). Because $\bar{\bar{\mu}}$ and $\bar{\mu}$ are isomorphisms, it follows from the five lemma that μ is also an isomorphism. ∎

There is also a functorial homomorphism

$$\mu: \mathrm{Hom}\,(A,G) \otimes \mathrm{Hom}\,(B,G') \to \mathrm{Hom}\,(A \otimes B, G \otimes G')$$

defined by $\mu(f \otimes f')(a \otimes b) = f(a) \otimes f'(b)$ for $f \in \mathrm{Hom}\,(A,G), f' \in \mathrm{Hom}\,(B,G')$, $a \in A$, and $b \in B$. In case $B = R$, $\mathrm{Hom}\,(B,G') \approx G'$, and μ corresponds to the homomorphism in lemma 5.

6 LEMMA *If B is a finitely generated free module, for arbitrary modules A and G, μ is an isomorphism*

$$\mu: \mathrm{Hom}\,(A,G) \otimes \mathrm{Hom}\,(B,R) \approx \mathrm{Hom}\,(A \otimes B, G)$$

PROOF The result is trivially true for $B = R$ and follows for a finite sum of copies of R because both sides commute with finite direct sums. ∎

7 COROLLARY *If A and B are free modules and either A and B or B and G' are finitely generated, μ is an isomorphism*

$$\mu: \mathrm{Hom}\,(A,G) \otimes \mathrm{Hom}\,(B,G') \approx \mathrm{Hom}\,(A \otimes B, G \otimes G')$$

PROOF Since A and B are free, so is $A \otimes B$. If A and B are finitely generated, so is $A \otimes B$, and there is a commutative diagram

$$\begin{tikzcd}
{[\mathrm{Hom}\,(R,G) \otimes \mathrm{Hom}\,(A,R)] \otimes [\mathrm{Hom}\,(R,G') \otimes \mathrm{Hom}\,(B,R)]} \arrow[r, "\bar{\mu}"] \arrow[d, "\mu \otimes \mu"'] & \mathrm{Hom}\,(R, G \otimes G') \otimes \mathrm{Hom}\,(A \otimes B, R) \arrow[d, "\mu"] \\
\mathrm{Hom}\,(A,G) \otimes \mathrm{Hom}\,(B,G') \arrow[r, "\mu"] & \mathrm{Hom}\,(A \otimes B, G \otimes G')
\end{tikzcd}$$

in which $\bar{\mu}((f_1 \otimes f_2) \otimes (f_3 \otimes f_4)) = \mu(f_1 \otimes f_3) \otimes \mu(f_2 \otimes f_4)$. By lemma 6, $\bar{\mu}$ is an isomorphism and so are both vertical maps. Therefore the bottom map is also an isomorphism.

If B and G' are finitely generated, there is a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}\,(A,G) \otimes \mathrm{Hom}\,(B,R) \otimes G' \arrow[r, "1 \otimes \mu"] \arrow[d, "\mu \otimes 1"'] & \mathrm{Hom}\,(A,G) \otimes \mathrm{Hom}\,(B,G') \arrow[d, "\mu"] \\
\mathrm{Hom}\,(A \otimes B, G) \otimes G' \arrow[r, "\mu"] & \mathrm{Hom}\,(A \otimes B, G \otimes G')
\end{tikzcd}$$

By lemma 5, both horizontal maps are isomorphisms, and by lemma 6, the left-hand vertical map is an isomorphism. Therefore the right-hand map is also an isomorphism. ∎

It follows from lemma 5 that if A is free and finitely generated, μ is an isomorphism

$$\mu\colon \operatorname{Hom}(A,R) \otimes A \approx \operatorname{Hom}(A,A)$$

The following lemma is a partial converse of this result.

8 LEMMA *If A is a module such that*

$$\mu\colon \operatorname{Hom}(A,R) \otimes A \to \operatorname{Hom}(A,A)$$

is an epimorphism, then A is finitely generated.

PROOF By hypothesis, there exist $f_i \in \operatorname{Hom}(A,R)$ and $a_i \in A$ for $1 \leq i \leq n$ such that $\mu(\Sigma f_i \otimes a_i) = 1_A$. Then, for any $a \in A$

$$a = \mu(\Sigma f_i \otimes a_i)(a) = \Sigma f_i(a)a_i$$

showing that A is generated by $\{a_i\}$. ∎

A graded module $\{C_q\}$ is said to be of *finite type* if C_q is finitely generated for every q. Thus a graded module C of finite type is finitely generated (as a graded module) if and only if $C_q = 0$, except for a finite set of integers q. The following lemma asserts that a chain complex whose homology is of finite type can be approximated by a chain complex of finite type.

9 LEMMA *Let C be a free chain complex such that $H(C)$ is of finite type. Then there is a free chain complex C' of finite type chain equivalent to C.*

PROOF For each q let F_q be a finitely generated submodule of $Z_q(C)$ such that F_q maps onto $H_q(C)$ under the epimorphism $Z_q(C) \to H_q(C)$. Let F'_q be the kernel of the epimorphism $F_q \to H_q(C)$. Define a chain complex $C' = \{C'_q,\partial'_q\}$ by $C'_q = F_q \oplus F'_{q-1}$ and $\partial'_q(c,c') = (c',0)$ for $c \in F_q$ and $c' \in F'_{q-1}$. Then C' is a free chain complex of finite type and $H_q(C') = F_q/F'_q \approx H_q(C)$. To define a chain equivalence $\tau\colon C' \to C$, choose for each q a homomorphism $\varphi_q\colon F'_q \to C_{q+1}$ such that $\partial_{q+1}\varphi_q(c') = c'$ for $c' \in F'_q$. Then define τ by $\tau(c,c') = c + \varphi_{q-1}(c')$ for $c \in F_q$ and $c' \in F'_{q-1}$. τ is a chain map and induces an isomorphism $\tau_*\colon H(C') \approx H(C)$. Because C' and C are both free, it follows from theorem 4.6.10 that τ is a chain equivalence. ∎

We are now ready for the universal-coefficient theorems toward which we have been heading.

10 THEOREM *Let C be a free chain complex and G be a module such that either $H(C)$ is of finite type or G is finitely generated. Then there is a functorial short exact sequence*

$$0 \to H^q(C;R) \otimes G \xrightarrow{\mu} H^q(C;G) \to H^{q+1}(C;R) * G \to 0$$

and this sequence is split.

PROOF If G is finitely generated, it follows from lemma 5 that

$$\mu\colon \operatorname{Hom}(C,R) \otimes G \approx \operatorname{Hom}(C,G)$$

Because Hom (C,R) is without torsion, Hom $(C,R) * G = 0$, and the result follows from theorem 5.4.1.

If $H(C)$ is of finite type, we use lemma 9 to replace C by a free chain complex C' of finite type. By corollary 7, μ: Hom $(C',R) \otimes G \approx$ Hom (C',G), and the result again follows for C' (and hence for C) from theorem 5.4.1. ∎

In a similar way we obtain the following *Künneth formula for cohomology.*

11 THEOREM *Let C and C' be nonnegative free chain complexes and G and G' be modules over a principal ideal domain such that $G * G' = 0$ and either $H(C)$ and $H(C')$ are of finite type or $H(C')$ is of finite type and G' is finitely generated. Then there is a functional short exact sequence*

$$0 \to [H^*(C;G) \otimes H^*(C';G')]^q \to H^q(C \otimes C';\, G \otimes G') \to [H^*(C;G) * H^*(C';G')]^{q+1} \to 0$$

and this sequence is split.

PROOF If $H(C)$ and $H(C')$ are of finite type, by lemma 9, we can replace C and C' by free chain complexes of finite type. Hence we are reduced to proving the result for the case where C and C' have finite type or where C' has finite type and G' is finitely generated. By corollary 7, there is an isomorphism μ: Hom $(C,G) \otimes$ Hom $(C',G') \approx$ Hom $(C \otimes C',\, G \otimes G')$. The result will now follow from theorem 5.4.2 as soon as we have verified that Hom $(C,G) *$ Hom (C',G') is acyclic.

We show that Hom $(C,G) *$ Hom $(C',G') = 0$. In case C and C' are both of finite type, Hom (C_p,G) is isomorphic to a finite direct sum of copies of G and Hom (C'_q,G') is isomorphic to a finite direct sum of copies of G'. Because $G * G' = 0$ by hypothesis, Hom $(C_p,G) *$ Hom $(C'_q,G') = 0$, and so Hom $(C,G) *$ Hom $(C',G') = 0$ in this case.

In case C' is of finite type, Hom (C'_q,G') is isomorphic to a finite direct sum of copies of G'. Hence it suffices to show that Hom $(C,G) * G' = 0$ if G' is finitely generated. Let

$$0 \to \bar{\bar{G}} \to \bar{G} \to G' \to 0$$

be a free resolution of G' with $\bar{G}$ finitely generated. Because $G * G' = 0$, there is a short exact sequence

$$0 \to G \otimes \bar{\bar{G}} \to G \otimes \bar{G} \to G \otimes G' \to 0$$

and a short exact sequence of cochain complexes (because C is free)

$$0 \to \operatorname{Hom}(C,\, G \otimes \bar{\bar{G}}) \to \operatorname{Hom}(C,\, G \otimes \bar{G}) \to \operatorname{Hom}(C,\, G \otimes G') \to 0$$

Using lemma 5, this implies the exactness of the sequence

$$0 \to \operatorname{Hom}(C,G) \otimes \bar{\bar{G}} \to \operatorname{Hom}(C,G) \otimes \bar{G} \to \operatorname{Hom}(C,G) \otimes G' \to 0$$

Hence Hom $(C,G) * G' = 0$, and so Hom $(C,G) *$ Hom $(C',G') = 0$ in either case. ∎

If A is a free finitely generated module, then

$$A \approx \operatorname{Hom}(\operatorname{Hom}(A,R), R)$$

Since Hom (A,R) is also free and finitely generated, it follows from corollary 7 that

$$A \otimes G \approx \operatorname{Hom}(\operatorname{Hom}(A,R), R) \otimes \operatorname{Hom}(R,G) \approx \operatorname{Hom}(\operatorname{Hom}(A,R), G)$$

We use this to express homology in terms of cohomology.

12 THEOREM *Let C be a free chain complex such that $H(C)$ is of finite type. For any module G there is a functorial short exact sequence*

$$0 \to \operatorname{Ext}(H^{q+1}(C;R), G) \to H_q(C;G) \xrightarrow{h} \operatorname{Hom}(H^q(C;R), G) \to 0$$

and this sequence is split.

PROOF By lemma 9, we are reduced to the case where C is of finite type. Then $C \otimes G \approx \operatorname{Hom}(\operatorname{Hom}(C,R), G)$, and the result follows, by theorem 3, on changing Hom (C,R) to a chain complex by changing the sign of the degree. ■

The following result is a version of lemma 8 valid for homology that is a partial converse to theorem 10.

13 THEOREM *Let C be a free chain complex such that for every module G the map μ: Hom $(C,R) \otimes G \to$ Hom (C,G) induces isomorphisms of all cohomology modules. Then $H_*(C)$ is of finite type.*

PROOF Because μ: $H^q(\operatorname{Hom}(C,R) \otimes H_q(C)) \approx H^q(\operatorname{Hom}(C,H_q(C)))$, it follows from theorem 3 that there exist $f_i \in \operatorname{Hom}(C_q,R)$ and $z_i \in H_q(C)$ such that $h\mu\{\sum f_i \otimes z_i\} = 1_{H_q(C)}$. Then, for any $z \in H_q(C)$ we have

$$z = \langle \mu\{\textstyle\sum f_i \otimes z_i\}, z\rangle = \textstyle\sum \langle f_i,z\rangle z_i$$

showing that $H_q(C)$ is generated by z_i. ■

Note that if the short exact sequence of theorem 10 is valid for a given C for all G, then the hypothesis of theorem 13 is satisfied, and so $H(C)$ is of finite type.

6 CUP AND CAP PRODUCTS

There is a cross product of cohomology classes from the tensor product of the cohomology of two spaces to the cohomology of their product space. By using the diagonal map of a space into its square, the cross product gives rise to a product in the cohomology module of a space. This multiplicative structure provides cohomology with more structure than just the essentially additive module structure. In this section we shall define these products and establish some of their elementary properties.

If $\{X \times B, A \times Y\}$ is an excisive couple in $X \times Y$, there is a *cohomology cross product*

$$\mu': H^p(X,A; G) \otimes H^q(Y,B; G') \to H^{p+q}((X,A) \times (Y,B); G \otimes G')$$

induced by the functorial homomorphism

$$\begin{array}{c} \mathrm{Hom}\,(\Delta(X)/\Delta(A),G) \otimes \mathrm{Hom}\,(\Delta(Y)/\Delta(B),G') \\ {}^{\mu}\downarrow \\ \mathrm{Hom}\,([\Delta(X)/\Delta(A)] \otimes [\Delta(Y)/\Delta(B)], G \otimes G') \end{array}$$

followed by an Eilenberg-Zilber cochain equivalence of the bottom module with $\mathrm{Hom}\,(\Delta(X \times Y)/\Delta(X \times B \cup A \times Y), G \otimes G')$. If $u \in H^p(X,A; G)$ and $v \in H^q(Y,B; G')$, we define

$$u \times v = \mu'(u \otimes v) \in H^{p+q}((X,A) \times (Y,B); G \otimes G')$$

From theorem 5.5.11 we obtain the following *Künneth formula for singular cohomology.*

1 THEOREM *Let $\{X \times B, A \times Y\}$ be an excisive couple in $X \times Y$ and let G and G' be modules such that $G * G' = 0$. If $H_*(X,A; R)$ and $H_*(Y,B; R)$ are of finite type or if $H_*(Y,B; R)$ is of finite type and G' is finitely generated, there is a functorial short exact sequence*

$$0 \to [H^*(X,A; G) \otimes H^*(Y,B; G')]^q \xrightarrow{\mu'} H^q((X,A) \times (Y,B); G \otimes G') \to$$
$$[H^*(X,A; G) * H^*(Y,B; G')]^{q+1} \to 0$$

and this sequence is split. ∎

The cohomology cross product satisfies the following analogues of statements 5.3.11 to 5.3.15.

2 *Let $f: (X,A) \to (X',A')$ and $g: (Y,B) \to (Y',B')$ be maps and let $u' \in H^p(X',A'; G)$ and $v' \in H^q(Y',B'; G')$. Then, in $H^{p+q}((X,A) \times (Y,B); G \otimes G')$, we have*

$$(f \times g)^*(u' \times v') = f^*u' \times g^*v' \quad ∎$$

3 *Let $p: (X,A) \times Y \to (X,A)$ be the projection to the first factor and let $\eta: G' \to H^*(Y;G')$ be the augmentation map. For $u \in H^q(X,A; G)$, in $H^q((X,A) \times Y; G \otimes G')$, we have*

$$p^*(\mu(u \otimes g')) = u \times \eta(g') \quad ∎$$

4 *For $u \in H^p(X,A; G)$, $v \in H^q(Y,B; G')$,* and *$w \in H^r(Z,C; G'')$, in $H^{p+q+r}((X,A) \times (Y,B) \times (Z,C); G \otimes G' \otimes G'')$, we have*

$$u \times (v \times w) = (u \times v) \times w \quad ∎$$

5 *Let $T: (X,A) \times (Y,B) \to (Y,B) \times (X,A)$ and $\varphi: G \otimes G' \to G' \otimes G$ interchange the factors. For $u \in H^p(X,A; G)$ and $v \in H^q(Y,B; G')$, in $H^{p+q}((X,A) \times (Y,B); G' \otimes G)$, we have*

$$T^*(v \times u) = (-1)^{pq}\varphi_*(u \times v) \quad \blacksquare$$

6 *Let $\{(X_1,A_1), (X_2,A_2)\}$ be an excisive couple of pairs in X and let $u \in H^p(X_1 \cap X_2, A_1 \cap A_2; G)$ and $v \in H^q(Y,B; G')$. For the connecting homomorphisms of appropriate Mayer-Vietoris sequences we have*

$$\delta^*(u \times v) = \delta^* u \times v$$

in $H^{p+q+1}((X_1 \cup X_2, A_1 \cup A_2) \times (Y,B); G \otimes G')$ and

$$\delta^*(v \times u) = (-1)^q v \times \delta^* u$$

in $H^{p+q+1}((Y,B) \times (X_1 \cup X_2, A_1 \cup A_2); G' \otimes G)$. $\blacksquare$

Consider the two functors $\Delta(X)$ and $\Delta(X) \otimes \Delta(X)$ on the category of topological spaces. Because $\Delta(X)$ is free with models $\{\Delta^q\}_{q\geq 0}$ and $\Delta(X) \otimes \Delta(X)$ is acyclic with models $\{\Delta^q\}_{q\geq 0}$ [that is, the reduced complex of $\Delta(\Delta^q) \otimes \Delta(\Delta^q)$ is acyclic for all q], it follows from the acyclic-model theorem 4.3.3 that there exist functorial chain maps τ_*: $\Delta(X) \to \Delta(X) \otimes \Delta(X)$ preserving augmentation, and any two are chain homotopic. Such a functorial chain map is called a *diagonal approximation.* The name stems from the fact that if τ'_X: $\Delta(X \times X) \to \Delta(X) \otimes \Delta(X)$ is a functorial chain equivalence given by the Eilenberg-Zilber theorem and d: $X \to X \times X$ is the diagonal map, then the composite

$$\Delta(X) \xrightarrow{\Delta(d)} \Delta(X \times X) \xrightarrow{\tau'_X} \Delta(X) \otimes \Delta(X)$$

is a diagonal approximation.

We construct a particular diagonal approximation called the *Alexander-Whitney diagonal approximation.* If σ: $\Delta^q \to X$ is a singular q-simplex, the *front i-face* ${}_i\sigma$ is defined for $0 \leq i \leq q$ to equal the composite $\sigma \circ \lambda$, where λ: $\Delta^i \to \Delta^q$ is the simplicial map defined by $\lambda(p_j) = p_j$ for $0 \leq j \leq i$. Similarly, the *back i-face* σ_i is defined for $0 \leq i \leq q$ to equal the composite $\sigma \circ \lambda'$, where λ': $\Delta^i \to \Delta^q$ is the simplicial map defined by $\lambda'(p_j) = p_{j+q-i}$ for $0 \leq j \leq i$. It is easy to verify that

$$\tau(\sigma) = \sum_{i+j=\deg \sigma} {}_i\sigma \otimes \sigma_j$$

defines a functorial chain map τ: $\Delta(X) \to \Delta(X) \otimes \Delta(X)$, and this chain map is the Alexander-Whitney diagonal approximation.

Let G and G' be R modules. A *pairing* of G and G' to an R module G'' is a homomorphism φ: $G \otimes G' \to G''$. For example, G and G' are always paired to $G \otimes G'$. Given such a pairing and given a diagonal approximation τ, there is a functorial cochain map

$$\bar\tau_X\colon \operatorname{Hom}(\Delta(X),G) \otimes \operatorname{Hom}(\Delta(X),G') \to \operatorname{Hom}(\Delta(X),G'')$$

defined to equal the composite

$$\operatorname{Hom}(\Delta(X),G) \otimes \operatorname{Hom}(\Delta(X),G') \xrightarrow{\mu}$$
$$\operatorname{Hom}(\Delta(X) \otimes \Delta(X), G \otimes G') \xrightarrow{\operatorname{Hom}(\tau_X,\varphi)} \operatorname{Hom}(\Delta(X),G'')$$

If $A \subset X$, then for $f \in \operatorname{Hom}(\Delta(X),G)$ and $f' \in \operatorname{Hom}(\Delta(X),G')$, we have

$$\bar{\tau}_X(f \otimes f') \mid \Delta(A) = \bar{\tau}_A(f \mid \Delta(A) \otimes f' \mid \Delta(A))$$

If $A_1, A_2 \subset X$ and f vanishes on A_1, f' vanishes on A_2, it follows that $\bar{\tau}_X(f \otimes f')$ vanishes on $\Delta(A_1) + \Delta(A_2)$. If $\{A_1,A_2\}$ is an excisive couple in X, it follows that $\bar{\tau}_X$ induces a homomorphism

$$H^p(X,A_1;\, G) \otimes H^q(X,A_2;\, G') \to H^{p+q}(X,\, A_1 \cup A_2;\, G'')$$

which is called the *cup-product homomorphism.* If $u \in H^p(X,A_1;\, G)$ and $v \in H^q(X,A_2;\, G')$, their cup product is denoted by

$$u \smile v \in H^{p+q}(X,A_1 \cup A_2;\, G'')$$

This product is a bilinear function of u and v and depends on the pairing φ but not on the particular diagonal approximation. The Alexander-Whitney diagonal approximation yields a particular map $\bar{\tau}$ which defines a cup product of cochains $f \smile f'$ for $f \in \operatorname{Hom}(\Delta_p(X),G)$ and $f' \in \operatorname{Hom}(\Delta_q(X),G')$ by

$$(f \smile f')(\sigma) = \varphi(f({}_p\sigma) \otimes f'(\sigma_q))$$

Then $\{f\} \smile \{f'\} = \{f \smile f'\}$ in $H^{p+q}(X,\, A_1 \cup A_2;\, G'')$.

As pointed out above, there exist diagonal approximations which are factored through $\Delta(d)$. This implies the following relation expressing the cup product in terms of the cross product.

7 THEOREM *If $\{X \times A_2,\, A_1 \times X\}$ is an excisive couple in $X \times X$, if $\{A_1,A_2\}$ is an excisive couple in X, and $\varphi\colon G \otimes G' \to G''$ is a pairing, then for $u \in H^p(X,A_1;\, G)$ and $v \in H^q(X,A_2;\, G')$, in $H^{p+q}(X,\, A_1 \cup A_2;\, G'')$, we have*

$$u \smile v = \varphi_*(d^*(u \times v)) \quad \blacksquare$$

The cup product has the following properties analogous to the corresponding properties of the cross product.

8 *Let $f\colon X \to Y$ map A_1 into B_1 and A_2 into B_2 and let $u \in H^p(Y,B_1;\, G)$ and $v \in H^q(Y,B_2;\, G')$. Let $f_1\colon (X,A_1) \to (Y,B_1)$, $f_2\colon (X,A_2) \to (Y,B_2)$, and $\bar{f}\colon (X,\, A_1 \cup A_2) \to (Y,\, B_1 \cup B_2)$ be maps defined by f. In $H^{p+q}(X,\, A_1 \cup A_2;\, G'')$, we have*

$$\bar{f}^*(u \smile v) = f_1^* u \smile f_2^* v \quad \blacksquare$$

9 *For any $u \in H^q(X,A;\, G)$ with the pairings $R \otimes G \approx G \approx G \otimes R$ we have*

$$1 \smile u = u = u \smile 1 \quad \blacksquare$$

10 *Given a commutative diagram, where φ, φ', ψ, and ψ' are pairings,*

$$\begin{array}{ccccc}
G_1 \otimes (G_2 \otimes G_3) \approx (G_1 \otimes G_2) \otimes G_3 & \xrightarrow{\varphi \otimes 1} & G_{12} \otimes G_3 \\
{\scriptstyle 1 \otimes \varphi'}\downarrow & & \downarrow{\scriptstyle \psi} \\
G_1 \otimes G_{23} & \xrightarrow{\psi'} & G_{123}
\end{array}$$

and given $u_1 \in H^p(X,A_1;\ G_1)$, $u_2 \in H^q(X,A_2;\ G_2)$, *and* $u_3 \in H^r(X,A_3;\ G_3)$, *then, in* $H^{p+q+r}(X,\ A_1 \cup A_2 \cup A_3;\ G_{123})$, *we have*

$$u_1 \smile (u_2 \smile u_3) = (u_1 \smile u_2) \smile u_3 \quad \blacksquare$$

11 *Given a commutative diagram of pairings*

$$\begin{tikzcd} G \otimes G' \arrow[rr, "\approx"] \arrow[dr] & & G' \otimes G \arrow[dl] \\ & G'' & \end{tikzcd}$$

and given $u \in H^p(X,A_1;\ G)$ *and* $v \in H^q(X,A_2;\ G')$, *in* $H^{p+q}(X,\ A_1 \cup A_2;\ G'')$, *we have*

$$u \smile v = (-1)^{pq} v \smile u \quad \blacksquare$$

12 *Let* $\{(X_1,A_1), (X_2,A_2)\}$ *be an excisive couple of pairs in X, let* $A \subset X_1 \cup X_2$, *and let* $i\colon (X_1 \cap X_2,\ A \cap X_1 \cap X_2) \subset (X_1 \cup X_2,\ A)$. *For elements* $u \in H^p(X_1 \cap X_2,\ A_1 \cap A_2;\ G)$ *and* $v \in H^q(X_1 \cup X_2,\ A;\ G')$ *and with the connecting homomorphisms of the appropriate Mayer-Vietoris sequences, in* $H^{p+q+1}(X_1 \cup X_2,\ A_1 \cup A_2 \cup A;\ G'')$, *we have*

$$\delta^*(u \smile i^*v) = \delta^* u \smile v$$
$$\delta^*(i^*v \smile u) = (-1)^q v \smile \delta^* u \quad \blacksquare$$

Let $\tau'\colon \Delta(X \times Y) \to \Delta(X) \otimes \Delta(Y)$ be a functorial chain equivalence given by the Eilenberg-Zilber theorem and let

$$T\colon [\Delta(X) \otimes \Delta(Y)] \otimes [\Delta(X) \otimes \Delta(Y)] \to [\Delta(X) \otimes \Delta(X)] \otimes [\Delta(Y) \otimes \Delta(Y)]$$

be the chain map defined by

$$T((c \otimes d) \otimes (c' \otimes d')) = (-1)^{\deg d \deg c'}(c \otimes c') \otimes (d \otimes d')$$

If τ is any diagonal approximation, it follows by the method of acyclic models that the diagram

$$\begin{tikzcd} \Delta(X \times Y) \arrow[r, "\tau_{X \times Y}"] \arrow[d, "\tau'"'] & \Delta(X \times Y) \otimes \Delta(X \times Y) \arrow[d, "T \circ (\tau' \otimes \tau')"] \\ \Delta(X) \otimes \Delta(Y) \arrow[r, "\tau_X \otimes \tau_Y"] & {[\Delta(X) \otimes \Delta(X)] \otimes [\Delta(Y) \otimes \Delta(Y)]} \end{tikzcd}$$

is chain homotopy commutative. This implies the following additional relation between cup products and cross products.

13 THEOREM *Let* $\varphi\colon G_1 \otimes G_2 \to G$ *and* $G_1' \otimes G_2' \to G'$ *be pairings and let* $G_1 \otimes G_1'$ *and* $G_2 \otimes G_2'$ *be paired to* $G \otimes G'$ *by the homomorphism*

$$(G_1 \otimes G_1') \otimes (G_2 \otimes G_2') \approx (G_1 \otimes G_2) \otimes (G_1' \otimes G_2') \xrightarrow{\varphi \otimes \varphi'} G \otimes G'$$

Given $u_1 \in H^p(X,A_1;\ G_1)$, $u_2 \in H^q(X,A_2;\ G_2)$, $v_1 \in H^r(Y,B_1\colon\ G_1')$, *and* $v_2 \in H^s(Y,B_2;\ G_2')$ *then with suitable excisiveness assumptions, we have, in* $H^{p+q+r+s}((X,\ A_1 \cup A_2) \times (Y,\ B_1 \cup B_2);\ G \otimes G')$,

$$(u_1 \times v_1) \smile (u_2 \times v_2) = (-1)^{qr}(u_1 \smile u_2) \times (v_1 \smile v_2) \quad \blacksquare$$

Combining theorem 13 with statements 3 and 9, we obtain the following result expressing the cross product in terms of the cup products.

14 COROLLARY *Let $\{X \times B, A \times Y\}$ be an excisive couple in $X \times Y$ and let p_1: $(X,A) \times Y \to (X,A)$ and p_2: $X \times (Y,B) \to (Y,B)$ be the projections. Given $u \in H^p(X,A;\, G)$ and $v \in H^q(Y,B;\, G')$, then, in $H^{p+q}((X,A) \times (Y,B);\, G \otimes G')$, we have*

$$u \times v = p_1^*(u) \smile p_2^*(v) \quad ■$$

With the last result we can give the following example of two polyhedra having isomorphic homology and cohomology modules but not isomorphic cup-product structures.

15 EXAMPLE Let p and q be integers ≥ 1 and let X be the space which is the union of S^p, S^q, and S^{p+q}, all identified at one point. If i: $S^p \subset X$, j: $S^q \subset X$, and k: $S^{p+q} \subset X$, then $i_* \tilde{H}(S^p) \oplus j_* \tilde{H}(S^q) \oplus k_* \tilde{H}(S^{p+q}) \approx \tilde{H}(X)$. Computing $H(S^p \times S^q)$ by the Künneth formula, we see that $H(X) \approx H(S^p \times S^q)$. By the universal-coefficient theorem, X and $S^p \times S^q$ have isomorphic homology and cohomology groups for any coefficient group. Since

$$k^*\colon H^{p+q}(X;\mathbf{Z}) \approx H^{p+q}(S^{p+q};\mathbf{Z})$$

and k^* commutes with the cup product, it follows that the cup product of integral cohomology classes of degrees p and q, respectively, in X is zero. However, it follows from corollary 14 that there are integral cohomology classes of $S^p \times S^q$ of degrees p and q, respectively, whose cup product is nonzero. Therefore $H^*(X;\mathbf{Z})$ and $H^*(S^p \times S^q;\, \mathbf{Z})$ are not isomorphic by an isomorphism of graded modules preserving the cup product. Hence X and $S^p \times S^q$ are not homeomorphic, nor even of the same homotopy type.

There is another product closely related to the cup product that multiplies homology and cohomology classes together. We begin with the observation that if C and C' are chain complexes and G and G' are paired to G'' by φ, there is a functorial homomorphism

$$h\colon \operatorname{Hom}(C',G) \otimes (C \otimes C' \otimes G') \to C \otimes G''$$

such that $h(f \otimes (c \otimes c' \otimes g')) = c \otimes \varphi(\langle f,c' \rangle \otimes g')$. A straightforward calculation shows that for $f \in \operatorname{Hom}(C'_q,G)$ and $\bar{c} \in (C \otimes C')_n \otimes G'$

$$\partial h(f \otimes \bar{c}) = (-1)^{n-q} h(\delta f \otimes \bar{c}) + h(f \otimes \partial \bar{c})$$

If X is a space and τ: $\Delta(X) \to \Delta(X) \otimes \Delta(X)$ is a diagonal approximation, a functorial map

$$\bar{\bar{\tau}}\colon \operatorname{Hom}(\Delta(X),G) \otimes (\Delta(X) \otimes G') \to \Delta(X) \otimes G''$$

is defined by $\bar{\bar{\tau}}(f \otimes c) = h(f \otimes \tau(c))$. The boundary formula yields

$$\partial \bar{\bar{\tau}}(f \otimes c) = (-1)^{\deg c - \deg f} \bar{\bar{\tau}}(\delta f \otimes c) + \bar{\bar{\tau}}(f \otimes \partial c)$$

Note that if A is a subset of X and $f \in \operatorname{Hom}(\Delta(X),G)$ vanishes on A, then for any $c \in \Delta(A) \otimes G'$, $\bar{\bar{\tau}}(f \otimes c) = 0$. It follows that if A_1, $A_2 \subset X$,

$f \in \text{Hom}(\Delta(X)/\Delta(A_1),G)$ is a cocycle, and $c \in \Delta(X) \otimes G'$ is a chain such that $\partial c \in [\Delta(A_1) + \Delta(A_2)] \otimes G'$, then $\bar{\bar{\tau}}(f \otimes c)$ is a chain of $\Delta(X) \otimes G''$ whose boundary is in $\Delta(A_2) \otimes G''$ [because $\partial\bar{\bar{\tau}}(f \otimes c) = \tau(\bar{\bar{f}} \otimes \partial c)$]. Furthermore, if f is the coboundary of a cochain which vanishes on $\Delta(A_1)$ or if c equals a boundary modulo $[\Delta(A_1) + \Delta(A_2)] \otimes G'$, then $\bar{\bar{\tau}}(f \otimes c)$ is a boundary modulo $\Delta(A_2) \otimes G''$. Hence $\bar{\bar{\tau}}$ defines a homomorphism [sending $\{f\} \otimes \{c\}$ to $\{\bar{\bar{\tau}}(f \otimes c)\}$]

$$H^q(X,A_1;\ G) \otimes H_n(\Delta(X)/[\Delta(A_1) + \Delta(A_2)];\ G') \to H_{n-q}(X,A_2;\ G'')$$

If $\{A_1,A_2\}$ is an excisive couple in X, this yields a homomorphism

$$H^q(X,A_1;\ G) \otimes H_n(X,\ A_1 \cup A_2;\ G') \to H_{n-q}(X,A_2;\ G'')$$

called the *cap product.* If $u \in H^q(X,A_1;\ G)$ and $z \in H_n(X,\ A_1 \cup A_2;\ G')$, their cap product is denoted by $u \frown z \in H_{n-q}(X,A_2;\ G'')$. It depends on the pairing φ but not on the particular diagonal approximation used to define $\bar{\bar{\tau}}$. The Alexander-Whitney diagonal approximation yields a map $\bar{\bar{\tau}}$ which defines a cap product on cochains and chains, denoted by $f \frown c$, by the formula

$$f \frown c = f \frown (\sum_\sigma \sigma \otimes g'_\sigma) = \sum_\sigma {}_{n-q}\sigma \otimes \varphi(\langle f,\sigma_q\rangle \otimes g'_\sigma)$$

for $f \in \text{Hom}(\Delta_q(X),G)$ and $c = \Sigma_\sigma\, \sigma \otimes g'_\sigma \in \Delta_n(X) \otimes G'$. Then $\{f\} \frown \{c\} = \{f \frown c\}$.

The cap product has the following properties analogous to those of the cup product.

16 *Let f: $X \to Y$ map A_1 to B_1 and A_2 to B_2 and let $u \in H^q(Y,B_1;\ G)$ and $z \in H_n(X,\ A_1 \cup A_2;\ G')$. Let f_1: $(X,A_1) \to (Y,B_1)$, f_2: $(X,A_2) \to (Y,B_2)$, and $\bar{f}$: $(X,\ A_1 \cup A_2) \to (Y,\ B_1 \cup B_2)$ be maps defined by f. Then, in $H_{n-q}(Y,B_2;\ G'')$, we have*

$$f_{2*}(f_1^* u \frown z) = u \frown \bar{f}_* z \quad \blacksquare$$

17 *For any $z \in H_n(X,A;\ G)$ with the pairing $R \otimes G \approx G$*

$$1 \frown z = z \quad \blacksquare$$

18 *Given a commutative diagram, where φ, φ', ψ, and ψ' are pairings,*

$$\begin{array}{ccc} G_1 \otimes (G_2 \otimes G_3) \approx (G_1 \otimes G_2) \otimes G_3 & \xrightarrow{\varphi \otimes 1} & G_{12} \otimes G_3 \\ {\scriptstyle 1 \otimes \varphi'}\downarrow & & \downarrow{\scriptstyle \psi} \\ G_1 \otimes G_{23} & \xrightarrow{\psi'} & G_{123} \end{array}$$

for $u \in H^p(X,A_1;\ G_1)$, $v \in H^q(X,A_2;\ G_2)$, and $z \in H_n(X,\ A_1 \cup A_2 \cup A_3;\ G_3)$, then, in $H_{n-p-q}(X,A_3;\ G_{123})$, we have

$$u \frown (v \frown z) = (u \smile v) \frown z \quad \blacksquare$$

19 *Let $u \in H^q(X,A;\ G)$ and $z \in H_q(X,A;\ G')$ and let ε: $H_0(X;\ G \otimes G') \to G \otimes G'$ be the augmentation. Then, in $G \otimes G'$,*

$$\varepsilon(u \frown z) = \langle u,z \rangle \quad \blacksquare$$

20 *Let $\{(X_1,A_1), (X_2,A_2)\}$ be an excisive couple in X and let $A \subset X_1 \cup X_2$ and $i: (X_1 \cap X_2, A \cap X_1 \cap X_2) \subset (X_1 \cup X_2, A)$. For $u \in H^q(X_1 \cup X_2, A; G)$ and $z \in H_n(X_1 \cup X_2, A_1 \cup A_2 \cup A; G')$, with the connecting homomorphisms of the appropriate Mayer-Vietoris sequences, in $H_{n-q-1}(X_1 \cap X_2, A_1 \cap A_2; G'')$, we have*

$$\partial_*(u \frown z) = i^*u \frown \partial_* z \quad \blacksquare$$

21 *Let $u_1 \in H^p(X,A_1; G_1)$, $u_2 \in H^q(Y,B_1; G_2)$, $z_1 \in H_m(X, A_1 \cup A_2; G_1')$, and $z_2 \in H_n(X, B_1 \cup B_2; G_2')$, and let G_1 and G_1' be paired to G_1'', G_2 and G_2' be paired to G_2'', and $(G_1 \otimes G_2)$ and $(G_1' \otimes G_2')$ be compatibly paired to $G_1'' \otimes G_2''$. Then, in $H_{m+n-p-q}((X,A_2) \times (Y,B_2); G_1'' \otimes G_2'')$, we have*

$$(u_1 \times u_2) \frown (z_1 \times z_2) = (-1)^{p(n-q)}(u_1 \frown z_1) \times (u_2 \frown z_2) \quad \blacksquare$$

7 HOMOLOGY OF FIBER BUNDLES

Cup and cap products are used in this section to study the homology of fiber bundles. We shall show that in case the cohomology of the total space maps epimorphically onto the cohomology of each fiber, the homology (or cohomology) of the total space is isomorphic to the homology (or cohomology) of the product space of the base and the fiber. For orientable sphere bundles this leads to a proof of the exactness of the Thom-Gysin sequences, which will be applied in the next section to compute the cohomology rings of projective spaces.

We begin with some algebraic considerations. Let $M = \{M_q\}$ be a free finitely generated graded R module and let $M^* = \{M^q = \operatorname{Hom}(M_q,R)\}$. Let (X,A) be a topological pair and $f: X \to Y$ be a continuous map. Given a homomorphism (of degree 0) $\theta: M^* \to H^*(X,A; R)$, there are homomorphisms (of degree 0) for any R module G

$$\Phi: H(X,A; G) \to H(Y;G) \otimes M$$
$$\Phi^*: H^*(Y;G) \otimes M^* \to H^*(X,A; G)$$

defined by $\Phi(z) = \sum_i f_*(\theta(m_i^*) \frown z) \otimes m_i$, where $\{m_i\}$ is a basis of M and $\{m_i^*\}$ is the dual basis of M^* (Φ is uniquely defined by this formula), and $\Phi^*(u \otimes m^*) = f^*u \smile \theta(m^*)$.

1 LEMMA *With the notation above, if Φ is an isomorphism for $G = R$, then Φ and Φ^* are isomorphisms for all R modules G.*

PROOF For each i let c_i^* be a cocycle of $\operatorname{Hom}(\Delta(X)/\Delta(A);R)$ representing the class $\theta(m_i^*)$ and assume that m_i (and hence also m_i^* and c_i^*) have degree q_i. Let $\tau: \Delta(X)/\Delta(A) \to \Delta(Y) \otimes M$ be the homomorphism (of degree 0) defined by

$$\tau(c) = \sum_i \Delta(f)(c_i^* \frown c) \otimes m_i$$

An easy computation shows that τ is a chain map and that the induced homomorphisms

$$\tau_*: H_*(X,A;\, G) \to H_*(\Delta(Y) \otimes M;\, G) \approx H_*(Y;G) \otimes M$$
$$\tau^*: H^*(Y;G) \otimes M^* \approx H^*(\operatorname{Hom}(\Delta(Y) \otimes M,\, G)) \to H^*(X,A;\, G)$$

equal Φ and Φ^*, respectively. Since Φ is assumed to be an isomorphism for $G = R$, the chain map τ induces an isomorphism of homology. The universal-coefficient theorems for homology and cohomology then imply that Φ and Φ^* are isomorphisms for all G. ∎

A *fiber-bundle pair* with base space B consists of a *total pair* $(E,\dot{E})$, a *fiber pair* $(F,\dot{F})$, and a *projection* $p: E \to B$ such that there exists an open covering $\{V\}$ of B and for each $V \in \{V\}$ a homeomorphism $\varphi_V: V \times (F,\dot{F}) \to (p^{-1}(V),\, p^{-1}(V) \cap \dot{E})$ such that the composite

$$V \times F \xrightarrow{\varphi_V} p^{-1}(V) \xrightarrow{p} V$$

is the projection to the first factor. If $A \subset B$, we let $E_A = p^{-1}(A)$ and $\dot{E}_A = p^{-1}(A) \cap \dot{E}$, and if $b \in B$, then $(E_b,\dot{E}_b)$ is the *fiber pair over* b.

Following are some examples.

2 For a space B and pair $(F,\dot{F})$ the *product-bundle pair* consists of the total pair $B \times (F,\dot{F})$ with projection to the first factor.

3 Given a bundle projection $\dot{p}: \dot{E} \to B$ with compact fiber $\dot{F}$, let E be the mapping cylinder of $\dot{p}$ and $p: E \to B$ the canonical retraction. Then $(E,\dot{E})$ is the total pair of a fiber-bundle pair over B with fiber $(F,\dot{F})$, where F is the cone over $\dot{F}$, and projection p.

4 If ξ is a q-sphere bundle over B, then $(E_\xi,\dot{E}_\xi)$ is the total pair of a fiber-bundle pair over B with fiber (E^{q+1},S^q) and projection $p_\xi: E_\xi \to B$.

Given a fiber-bundle pair with total pair $(E,\dot{E})$ and fiber pair $(F,\dot{F})$, a *cohomology extension of the fiber* is a homomorphism $\theta: H^*(F,\dot{F};\, R) \to H^*(E,\dot{E};\, R)$ of graded modules (of degree 0) such that for each $b \in B$ the composite

$$H^*(F,\dot{F};\, R) \xrightarrow{\theta} H^*(E,\dot{E};\, R) \to H^*(E_b,\dot{E}_b;\, R)$$

is an isomorphism. The following statements are easily verified.

5 *Let $\bar{p}: B \times (F,\dot{F}) \to (F,\dot{F})$ be the projection to the second factor. Then*

$$\theta = \bar{p}^*: H^*(F,\dot{F};\, R) \to H^*(B \times (F,\dot{F});\, R)$$

is a cohomology extension of the fiber of the product-bundle pair. ∎

6 *Let $\theta: H^*(F,\dot{F};\, R) \to H^*(E,\dot{E};\, R)$ be a cohomology extension of the fiber of a fiber-bundle pair over B and let $f: B' \to B$ be a map. There is an induced bundle pair over B', with total pair $(E',\dot{E}')$ and fiber $(F,\dot{F})$, and there is a map*

$\bar{f}$: $(E',\dot{E}') \to (E,\dot{E})$ commuting with projections. Then the composite

$$H^*(F,\dot{F};\,R) \xrightarrow{\theta} H^*(E,\dot{E};\,R) \xrightarrow{\bar{f}^*} H^*(E',\dot{E}';\,R)$$

is a cohomology extension of the fiber in the induced bundle. ∎

7 *Given a fiber-bundle pair over B with total pair $(E,\dot{E})$, let the path components of B be $\{B_j\}$ and let $(E_j,\dot{E}_j)$ be the induced total pair over B_j. A cohomology extension θ of the fiber of the bundle pair over B corresponds to a family of cohomology extensions $\{\theta_j\}$ of the induced bundle pairs over B_j.* ∎

We now establish the local form of the theorem toward which we are heading. It shows that any cohomology extension of the fiber in a product-bundle pair has homology properties as nice as the one given in statement 5 above.

8 LEMMA *Let $(F,\dot{F})$ be a pair such that $H_*(F,\dot{F};\,R)$ is free and finitely generated over R and let θ: $H^*(F,\dot{F};\,R) \to H^*(B \times (F,\dot{F});\,R)$ be a cohomology extension of the fiber of the product-bundle pair. Then the homomorphisms*

$$\Phi\colon H_*(B \times (F,\dot{F});\,G) \to H_*(B;G) \otimes H_*(F,\dot{F};\,R)$$
$$\Phi^*\colon H^*(B;G) \otimes H^*(F,\dot{F};\,R) \to H^*(B \times (F,\dot{F});\,G)$$

are isomorphisms for all R modules G.

PROOF By lemma 1, it suffices to prove that Φ is an isomorphism for $G = R$. If $\{B_j\}$ is the set of path components of B, then

$$H_*(B \times (F,\dot{F});\,R) \approx \bigoplus_j H_*(B_j \times (F,\dot{F});\,R)$$

and

$$H_*(B;R) \otimes H_*(F,\dot{F};\,R) \approx \bigoplus_j H_*(B_j;R) \otimes H_*(F,\dot{F};\,R)$$

Therefore it suffices to prove the result for a path-connected space B. For such a B, $R \approx H^0(B;R)$.

By the Künneth formula, $H_*(B \times (F,\dot{F});\,R) \approx H_*(B;R) \otimes H_*(F,\dot{F};\,R)$. We define graded submodules N_s of $H_*(B;R) \otimes H_*(F,\dot{F};\,R)$ by

$$(N_s)_q = \bigoplus_{i+j=q,\, j\geq s} H_i(B;R) \otimes H_j(F,\dot{F};\,R)$$

Then

$$H_*(B;R) \otimes H_*(F,\dot{F};\,R) = N_0 \supset N_1 \supset \cdots \supset N_s \supset N_{s+1}$$

and $N_s = 0$ for large enough s. If $u \in H^s(F,\dot{F};\,R)$, then $\theta(u) = 1 \times \lambda(u) + \bar{u}$, where $\bar{u} \in \bigoplus_{i+j=s,j<s} H^i(B;R) \otimes H^j(F,\dot{F};\,R)$ and $\theta(u) \mid [b \times (F,\dot{F})] = 1 \times \lambda(u)$. Because θ is a cohomology extension of the fiber, λ is an automorphism of $H^*(F,\dot{F};\,R)$. Let $z' \in H_s(F,\dot{F};\,R)$ and consider $z \times z' \in N_s$. Then

$$\Phi(z \times z') = \sum_i p_*(\theta(m_i^*) \frown (z \times z')) \otimes m_i$$

and if $\deg m_i < s$, then $\theta(m_i^*) \frown (z \times z') \in N_1$ and $p_*(N_1) = 0$. Therefore

$\Phi(z \times z') \in N_s$, and so Φ maps N_s into itself for all s. Because of the short exact sequences

$$0 \to N_{s+1} \to N_s \to N_s/N_{s+1} \to 0$$

and the five lemma, it follows by downward induction on s that Φ is an isomorphism if and only if it induces an isomorphism of N_s/N_{s+1} onto itself for all s. For $z' \in H_s(F,\dot{F};\, R)$, computing $\Phi(z \times z')$ in N_s/N_{s+1}, we obtain

$$\begin{aligned}\Phi(z \times z') &= \sum_{\deg m_i \geq s} p_*[(1 \times \lambda(m_i^*) + \bar{m}_i^*) \frown (z \times z')] \otimes m_i \\ &= \sum_{\deg m_i = s} p^*[1 \times \lambda(m_i^*) \frown (z \times z')] \otimes m_i\end{aligned}$$

because $\bar{m}_i^* \frown (z \times z') \in N_1$ and $p_*(N_1) = 0$. Now, by properties 5.6.21, 5.6.19, and 5.6.17,

$$\sum_{\deg m_i = s} p_*[1 \times \lambda(m_i^*) \frown (z \times z')] \otimes m_i = \sum_{\deg m_i = s} z \otimes \langle \lambda(m_i^*), z' \rangle m_i = z \otimes \lambda_*(z')$$

where $\lambda_*: H_*(F,\dot{F};\, R) \to H_*(F,\dot{F};\, R)$ is the automorphism dual to λ. Hence $\Phi(z \times z') = z \times \lambda_*(z')$ in N_s/N_{s+1}, showing that Φ induces an isomorphism of N_s/N_{s+1} for all s. ∎

The following *Leray-Hirsch theorem* shows that fiber-bundle pairs with cohomology extensions of the fiber have homology and cohomology modules isomorphic to those of the product of the fiber pair and the base.

9 THEOREM *Let $(E,\dot{E})$ be the total pair of a fiber-bundle pair with base B and fiber pair $(F,\dot{F})$. Assume that $H_*(F,\dot{F};\, R)$ is free and finitely generated over R and that θ is a cohomology extension of the fiber. Then the homomorphisms*

$$\Phi: H_*(E,\dot{E};\, G) \to H_*(B;G) \otimes H_*(F,\dot{F};\, R) \qquad \Phi(z) = \sum_i p_*(\theta(m_i^*) \frown z) \otimes m_i$$

$$\Phi^*: H^*(B;G) \otimes H^*(F,\dot{F};\, R) \to H^*(E,\dot{E};\, G) \qquad \Phi^*(u \otimes v) = p^*(u) \smile \theta(v)$$

are isomorphisms (of graded modules) for all R modules G.

PROOF By lemma 1, it suffices to prove the result for the map Φ in the case $G = R$. For any subset $A \subset B$ let θ_A be the composite

$$H^*(F,\dot{F};\, R) \xrightarrow{\theta} H^*(E,\dot{E};\, R) \to H^*(E_A,\dot{E}_A;\, R)$$

Then θ_A is a cohomology extension of the fiber in the induced bundle over A. It follows from lemma 8 that if the induced bundle over A is homeomorphic to the product-bundle pair $A \times (F,\dot{F})$, then

$$\Phi_A: H_*(E_A,\dot{E}_A;\, R) \approx H_*(A;R) \otimes H_*(F,\dot{F};\, R)$$

Hence Φ_V is an isomorphism for all sufficiently small open sets V.

If V and V' are open sets in B, then $\{(E_V,\dot{E}_V), (E_{V'},\dot{E}_{V'})\}$ is an excisive couple of pairs in E, and it follows from property 5.6.20 that Φ_V, $\Phi_{V'}$, $\Phi_{V\cap V'}$, and $\Phi_{V\cup V'}$ map the exact Mayer-Vietoris sequence of $(E_V,\dot{E}_V)$ and $(E_{V'},\dot{E}_{V'})$ into the tensor product of the exact Mayer-Vietoris sequence of V and V' by $H_*(F,\dot{F};\ R)$. Since $H_*(F,\dot{F};\ R)$ is free over R, its tensor product with any exact sequence is exact. Therefore, if Φ_V, $\Phi_{V'}$, and $\Phi_{V\cap V'}$ are isomorphisms, it follows from the five lemma that $\Phi_{V\cup V'}$ is also an isomorphism. By induction, Φ_U is an isomorphism for any U which is a finite union of sufficiently small open sets. Let $\mathcal{U}$ be the collection of these sets. Since any compact subset of B lies in some element of $\mathcal{U}$, $H_*(B;R) \approx \lim_{\rightarrow} \{H_*(U;R)\}_{U\in\mathcal{U}}$. Also, any compact subset of E lies in E_U for some $U \in \mathcal{U}$, so $H_*(E,\dot{E};\ R) \approx \lim_{\rightarrow} \{H_*(E_U,\dot{E}_U;\ R)\}$. Because the tensor product commutes with direct limits and Φ corresponds to $\lim_{\rightarrow} \{\Phi_U\}_{U\in\mathcal{U}}$ under these isomorphisms, Φ is also an isomorphism. ■

The above argument proves directly that Φ is an isomorphism for any coefficient module G. A similar argument does not appear possible for Φ^*, because it is not true that $H^*(B;R)$ is isomorphic to the inverse limit $\lim_{\leftarrow} \{H^*(U;R)\}_{U\in\mathcal{U}}$. It should be noted that in theorem 9 we have said nothing about commutativity of Φ^* with cup products, because it is not true, in general, that Φ^* preserves cup products.

We now specialize to the case of sphere bundles. Because

$$H^r(E^{q+1},S^q;\ R) \approx \begin{cases} 0 & r \neq q+1 \\ R & r = q+1 \end{cases}$$

if ξ is a q-sphere bundle, a cohomology extension of the fiber in ξ is an element $U \in H^{q+1}(E_\xi,\dot{E}_\xi;\ R)$ such that for any $b \in B$, the restriction of U to $(p^{-1}(b),\ p^{-1}(b) \cap \dot{E})$ is a generator of $H^{q+1}(p^{-1}(b),\ p^{-1}(b) \cap \dot{E};\ R)$. Such a cohomology class is called an *orientation class* (*over* R) of the bundle. If orientations of the bundle exist, the bundle is called *orientable*. An *oriented sphere bundle* is a pair (ξ,U_ξ) consisting of a sphere bundle ξ and an orientation class of U_ξ of ξ.

If U is an orientation class of ξ over $\mathbf{Z}$ and if 1 is the unit element of R, then $\mu(U \otimes 1)$ is an orientation class of ξ over R. Therefore a sphere bundle orientable over $\mathbf{Z}$ is orientable over any R.

If (ξ,U_ξ) is an oriented sphere bundle over B and $f\colon B' \to B$, then $(f^*\xi,\bar{f}^*U_\xi)$ is an oriented sphere bundle over B' [where $\bar{f}\colon (E_{f^*\xi},\dot{E}_{f^*\xi}) \to (E_\xi,\dot{E}_\xi)$ is associated to f].

From theorem 9 we get the following *Thom isomorphism theorem.*

10 THEOREM *Let (ξ,U_ξ) be an oriented q-sphere bundle over B. There are natural isomorphisms for any R module G*

$$\Phi_\xi\colon H_n(E_\xi,\dot{E}_\xi;\ G) \xrightarrow{\approx} H_{n-q-1}(B;G) \qquad \Phi_\xi(z) = p_*(U_\xi \frown z)$$

$$\Phi_\xi^*\colon H^r(B;G) \xrightarrow{\approx} H^{r+q+1}(E_\xi,\dot{E}_\xi;\ G) \qquad \Phi_\xi^*(v) = p^*v \smile U_\xi$$

PROOF Let m and m^* be dual generators of $H_{q+1}(E^{q+1},S^q;\ R)$ and $H^{q+1}(E^{q+1},S^q;\ R)$, respectively, and define a cohomology extension θ by $\theta(m^*) = U_\xi$. Then Φ_ξ is the composite

$$H_n(E_\xi,\dot{E}_\xi;\ G) \xrightarrow{\Phi} H_{n-q-1}(B;G) \otimes H_{q+1}(E^{q+1},S^q;R) \approx H_{n-q-1}(B;G)$$

where the second map sends $z \otimes m$ to z. By theorem 9, Φ is an isomorphism, and so Φ_ξ is an isomorphism. A similar argument shows that Φ_ξ^* is an isomorphism. These isomorphisms are natural for induced bundles because of naturality properties of the cup and cap products. ∎

This result implies the exactness of the following *Thom-Gysin sequences* of a sphere bundle.

11 THEOREM *Let (ξ, U_ξ) be an oriented q-sphere bundle with base B and projection $\dot p = p \mid \dot E$: $\dot E \to B$. For any R module G there are natural exact sequences*

$$\cdots \to H_n(\dot E_\xi;G) \xrightarrow{\dot p_*} H_n(B;G) \xrightarrow{\Psi_\xi} H_{n-q-1}(B;G) \xrightarrow{\rho} H_{n-1}(\dot E_\xi;G) \to \cdots$$
$$\cdots \to H^r(B;G) \xrightarrow{\dot p^*} H^r(\dot E_\xi;G) \xrightarrow{\rho^*} H^{r-q}(B;G) \xrightarrow{\Psi_\xi^*} H^{r+1}(B;G) \to \cdots$$

in which Ψ_ξ and Ψ_ξ^* have properties

$$\Psi_\xi(v \frown z) = (-1)^{(q+1)\deg v}\, \Psi_\xi^*(v) \frown z$$
$$\Psi_\xi^*(v_1 \smile v_2) = v_1 \smile \Psi_\xi^*(v_2)$$

PROOF There is a commutative diagram (with any coefficient module)

$$\begin{array}{ccccccccc}
\cdots \to & H_n(\dot E) & \xrightarrow{i_*} & H_n(E) & \xrightarrow{j_*} & H_n(E,\dot E) & \xrightarrow{\partial} & H_{n-1}(\dot E) & \to \cdots \\
& & \dot p_* \searrow & \approx \downarrow p_* & & \approx \downarrow \Phi_\xi & & & \\
& & & H_n(B) & & H_{n-q-1}(B) & & &
\end{array}$$

the top row of which is exact. Since p is a deformation retraction of E onto B, p_* is an isomorphism. By theorem 10, Φ_ξ is an isomorphism. The desired sequence is obtained by defining $\Psi_\xi = \Phi_\xi j_* p_*^{-1}$ and $\rho = \partial \Phi_\xi^{-1}$. Similarly, the cohomology sequence is defined by $\Psi_\xi^* = p^{*-1} j^* \Phi_\xi^*$ and $\rho^* = \Phi_\xi^{*-1}\delta$. We verify the formula for Ψ_ξ.

$$\begin{aligned}
\Psi_\xi(v \frown z) &= \Phi_\xi j_* p_*^{-1}(v \frown z) = \Phi_\xi j_*(p^*(v) \frown p_*^{-1}(z)) \\
&= \Phi_\xi(p^*(v) \frown j_* p_*^{-1}(z)) = p_*(U \frown [p^*(v) \frown j_* p_*^{-1}(z)]) \\
&= p_*(j^*[U \smile p^*(v)] \frown p_*^{-1}(z)) \\
&= (-1)^{(q+1)\deg v} p_*[j^* \Phi_\xi^*(v) \frown p_*^{-1}(z)] \\
&= (-1)^{(q+1)\deg v}\, \Psi_\xi^*(v) \frown z \quad ∎
\end{aligned}$$

Note that the isomorphisms Φ and Φ^* of the Thom isomorphism theorem depend on the choice of the orientation class U of the bundle. Therefore the homomorphisms ρ and Ψ and ρ^* and Ψ^* of the Thom-Gysin sequences also depend on the orientation class. In case B is path connected and U and U' are orientation classes of a sphere bundle over B, it follows from theorem 10 that there is an element $r \in R$ such that

$$U' = p^*(r \times 1) \smile U = r[p^*(1) \smile U]$$

If $b_0 \in B$, then

$$U' \mid (p^{-1}(b_0), p^{-1}(b_0) \cap \dot E) = r[U \mid (p^{-1}(b_0), p^{-1}(b_0) \cap \dot E)]$$

Therefore we have the next result.

12 LEMMA *Two orientation classes U and U' of a sphere bundle over a path-connected base space B are equal if and only if for some $b_0 \in B$*

$$U \mid (p^{-1}(b_0), p^{-1}(b_0) \cap \dot{E}) = U' \mid (p^{-1}(b_0), p^{-1}(b_0) \cap \dot{E}) \quad \blacksquare$$

If B is not path connected, let $\{B_j\}$ be the set of path components of B and let $(E_j,\dot{E}_j)$ be the part of $(E,\dot{E})$ over B_j. Then

$$H^*(E,\dot{E};\, R) \approx \times_j\, H^*(E_j,\dot{E}_j;\, R)$$

and we also obtain the following result.

13 LEMMA *Two orientation classes U and U' of a sphere bundle with base space B are equal if and only if for all $b \in B$*

$$U \mid (p^{-1}(b), p^{-1}(b) \cap \dot{E}) = U' \mid (p^{-1}(b), p^{-1}(b) \cap \dot{E}) \quad \blacksquare$$

In case $R = \mathbf{Z}_2$, then $H^{q+1}(p^{-1}(b), p^{-1}(b) \cap \dot{E};\, \mathbf{Z}_2) \approx \mathbf{Z}_2$ for all $b \in B$. Therefore this module has a unique nonzero element, and we obtain the following consequence of lemma 13.

14 COROLLARY *Any two orientation classes over $\mathbf{Z}_2$ of a sphere bundle are equal.* $\blacksquare$

Thus, for $R = \mathbf{Z}_2$ the homomorphisms Φ, ρ, and Ψ and Φ^*, ρ^*, and Ψ^* are all unique.

The *characteristic class* Ω_ξ of an oriented q-sphere bundle (ξ, U_ξ) is defined to be the element

$$\Omega_\xi = \Psi_\xi^*(1) \in H^{q+1}(B;R)$$

This is functorial (that is, $\Omega_{f^*\xi} = f^*\Omega_\xi$). From the multiplicative properties of Ψ_ξ and Ψ_ξ^* in theorem 11 we obtain the following equations.

15 *For $z \in H_n(B;G)$*

$$\Psi_\xi(z) = \Omega_\xi \frown z$$

and for $v \in H^r(B;G)$

$$\Psi_\xi^*(v) = v \smile \Omega_\xi \quad \blacksquare$$

We now investigate the existence of orientation classes for a sphere bundle. Let (X,X') be a pair and let $\{A_j\}_{j \in J}$ be an indexed collection of subsets $A_j \subset X$. An indexed collection

$$\{u_j \in H^n(A_j, A_j \cap X';\, G)\}_{j \in J}$$

is said to be *compatible* if for all $j, j' \in J$

$$u_j \mid (A_j \cap A_{j'}, A_j \cap A_{j'} \cap X') = u_{j'} \mid (A_j \cap A_{j'}, A_j \cap A_{j'} \cap X')$$

The compatible collections $\{u_j\}$ constitute an R module $H^n(\{A_j\},X';\, G)$. Clearly, the restriction maps

$$H^n(X,X';\, G) \to H^n(A_j, A_j \cap X';\, G)$$

define a natural homomorphism $H^n(X,X'; G) \to H^n(\{A_j\},X'; G)$.

16 LEMMA *Let $(E,\dot E)$ be a fiber-bundle pair with base B, projection $p: E \to B$, and fiber pair $(F,\dot F)$. Assume that for some $n > 0$, $H_i(F,\dot F; R) = 0$ for $i < n$. Then*

(a) For all $A \subset B$ and all R modules G

$$H_i(p^{-1}(A), p^{-1}(A) \cap \dot E; G) = 0 = H^i(p^{-1}(A), p^{-1}(A) \cap \dot E; G) \qquad i < n$$

(b) If $\{V\}$ is any open covering of B, then in degree n the natural homomorphism is an isomorphism

$$H^n(E,\dot E; G) \approx H^n(\{p^{-1}V\},\dot E; G)$$

PROOF By the universal-coefficient formula, it suffices to prove (*a*) for $G = R$. If $A \subset B$ is such that $(p^{-1}(A), p^{-1}(A) \cap \dot E)$ is homeomorphic to $A \times (F,\dot F)$, then by the Künneth formula,

$$H_i(p^{-1}(A), p^{-1}(A) \cap \dot E; R) \approx H_i(A \times (F,\dot F); R) = 0 \qquad i < n$$

From this it follows (as in the proof of theorem 9) by induction on the number of coordinate neighborhoods of the bundle needed to cover A (using the Mayer-Vietoris sequence and the five lemma) that (*a*) holds for all compact $A \subset B$. By taking direct limits, (*a*) holds for any A.

For (*b*), let $\{W\}$ be the collection of finite unions of elements of $\{V\}$. By (*a*) and the universal-coefficient formula for cohomology, there is a commutative diagram

$$\begin{array}{ccc} H^n(E,\dot E; G) & \approx & \operatorname{Hom}(H_n(E,\dot E;R), G) \\ \downarrow & & \downarrow\approx \\ \lim_{\leftarrow}\{H^n(p^{-1}(W), p^{-1}(W) \cap \dot E; G)\} & \approx & \lim_{\leftarrow}\{\operatorname{Hom}(H_n(p^{-1}(W), p^{-1}(W) \cap \dot E; R),G)\} \end{array}$$

Hence we need only prove that a compatible collection $\{u_V\}_{V\in\{V\}}$ extends to a unique compatible collection $\{u_W\}_{W\in\{W\}}$. This follows by using Mayer-Vietoris sequences again and from the fact that $H^i(p^{-1}(W), p^{-1}(W) \cap \dot E; G) = 0$ for $i < n$. ∎

For sphere bundles we have the following immediate consequence.

17 COROLLARY *A sphere bundle ξ with base B is orientable if and only if there is a covering $\{V\}$ of B and a compatible family $\{u_V\}$, where u_V is an orientation class of $\xi \mid V$ for each $V \in \{V\}$.* ∎

Since a trivial sphere bundle is orientable, corollaries 17 and 14 imply the following result.

18 COROLLARY *Any sphere bundle has a unique orientation class over $\mathbf{Z}_2$.* ∎

By theorem 2.8.12, there is a contravariant functor from the fundamental groupoid of the base space B of a sphere bundle ξ to the homotopy category which assigns to $b \in B$ the fiber pair $(E_b,\dot E_b)$ over b and to a path class $[\omega]$ in B a homotopy class $h[\omega] \in [E_{\omega(0)},\dot E_{\omega(0)}; E_{\omega(1)},\dot E_{\omega(1)}]$. For fixed R there is then a covariant functor from the fundamental groupoid of B to the category of R modules which assigns to $b \in B$ the module $H^{q+1}(E_b,\dot{E}_b;\, R)$ and to a path class $[\omega]$ the homomorphism

$$h[\omega]^*\colon H^{q+1}(E_{\omega(1)},\dot{E}_{\omega(1)};\, R) \to H^{q+1}(E_{\omega(0)},\dot{E}_{\omega(0)};\, R)$$

19 THEOREM *A sphere bundle ξ is orientable over R if and only if for every closed path ω in B, $h[\omega]^* = 1$.*

PROOF If ξ is orientable with orientation class $U \in H^{q+1}(E,\dot{E};\, R)$, for any small path ω in B (and hence for any path)

$$h[\omega]^*(U \mid (E_{\omega(1)},\dot{E}_{\omega(1)})) = U \mid (E_{\omega(0)},\dot{E}_{\omega(0)})$$

Since $U \mid (E_b,\dot{E}_b)$ is a generator of $H^{q+1}(E_b,\dot{E}_b;\, R)$, this implies that $h[\omega]^* = 1$ for any closed path ω.

Conversely, if $h[\omega]^* = 1$ for every closed path ω in B, there exist generators $U_b \in H^{q+1}(E_b,\dot{E}_b;\, R)$ such that for any path class $[\omega]$ in B, $h[\omega]^*(U_{\omega(1)}) = U_{\omega(0)}$. If V is any subset of B such that $\xi \mid V$ is trivial, it is easy to see that there is an orientation class U_V of $\xi \mid V$ such that $U_V \mid (E_b,\dot{E}_b) = U_b$ for all $b \in V$. If $\{V\}$ is an open covering of B by sets such that $\xi \mid V$ is trivial for all V, then $\{U_V\}$ is a compatible family of orientations, and by corollary 17, ξ is orientable. ∎

20 COROLLARY *A sphere bundle with a simply connected base is orientable over any R.* ∎

8 THE COHOMOLOGY ALGEBRA

The cup product in cohomology makes the cohomology (over R) of a topological pair a graded R algebra. In the first part of this section we define the relevant algebraic concepts and compute this algebra over $\mathbf{Z}_2$ for a real projective space and over any R for complex and quaternionic projective space. This is applied to prove the Borsuk-Ulam theorem.

For the case of an H space, there is even more algebraic structure that can be introduced in the cohomology algebra. The cohomology of such a space is a Hopf algebra, and the second part of the section is devoted to its definition and some results about its structure. The section concludes with a proof of the Hopf theorem about the cohomology algebra of a compact connected H space.

A *graded R algebra* consists of a graded R module $A = \{A^q\}$ and a homomorphism of degree 0

$$\mu\colon A \otimes A \to A$$

called the *product* of the algebra (μ then maps $A^p \otimes A^q$ into A^{p+q} for all p and q). For $a, a' \in A$ we write $aa' = \mu(a \otimes a')$. The product is *associative* if $(aa')a'' = a(a'a'')$ for all a, a', $a'' \in A$ and is *commutative* if $aa' = (-1)^{\deg a \deg a'} a'a$ for all $a, a' \in A$.

1 EXAMPLE If (X,A) is a topological pair, then $H^*(X,A;\ R)$ is a graded R algebra whose product is the cup product (with respect to the multiplication pairing of R with itself to R). It follows from property 5.6.10 that this product is associative and from property 5.6.11 that it is commutative. If $A = \varnothing$, it follows from property 5.6.9 that 1 is a unit element of the algebra $H^*(X;R)$. $H^*(X,A;\ R)$ is called the *cohomology algebra* of (X,A) over R.

2 EXAMPLE The *polynomial algebra over R generated by x of degree* $n > 0$, denoted by $S_n(x)$, is defined by

$$[S_n(x)]^q = \begin{cases} 0 & q \not\equiv 0\ (n) \text{ or } q < 0 \\ \text{free } R \text{ module generated by } x_p & q = pn,\ p \geq 0 \end{cases}$$

with the product $(\alpha x_p)(\beta x_q) = (\alpha\beta)x_{p+q}$ for $\alpha,\ \beta \in R$. It is then clear that x_0 is a unit element and that $x_p = (x_1)^p$. If we denote x_1 by x, then $x_p = x^p$. Thus, disregarding the graded structure, $S_n(x)$ is simply the polynomial algebra over R in one indeterminate x. The *truncated polynomial algebra over R generated by x of degree n and height h*, denoted by $T_{n,h}(x)$, is defined to be the quotient of $S_n(x)$ by the graded ideal generated by x^h. If $h = 2$, this is called the *exterior algebra generated by x of degree n* and is denoted by $E_n(x)$.

If A and B are graded R algebras, their tensor product $A \otimes B$ is also a graded R algebra with product

$$(a \otimes b)(a' \otimes b') = (-1)^{\deg b \deg a'} aa' \otimes bb'$$

If A and B have associative or commutative products, so does $A \otimes B$.

3 EXAMPLE If R is a field and (X,A) and (Y,B) are topological pairs such that either $H_*(X,A;\ R)$ or $H_*(Y,B;\ R)$ is of finite type, it follows from theorem 5.5.11 that

$$H^*(X,A;\ R) \otimes H^*(Y,B;\ R) \approx H^*((X,A) \times (Y,B);\ R)$$

We compute the graded $\mathbf{Z}_2$ algebra $H^*(P^n;\mathbf{Z}_2)$ for real projective space P^n. Note that the double covering $p\colon S^n \to P^n$ is a 0-sphere bundle. We let $w_n \in H^1(P^n;\mathbf{Z}_2)$ be the characteristic class (over $\mathbf{Z}_2$) of this bundle.

4 THEOREM *For $n \geq 1$, $H^*(P^n;\mathbf{Z}_2)$ is a truncated polynomial algebra over $\mathbf{Z}_2$ generated by w_n of degree 1 and height $n + 1$.*

PROOF All coefficients in the proof will be $\mathbf{Z}_2$ and will be omitted. By corollary 5.7.18 and theorem 5.7.11, there is an exact Thom-Gysin sequence

$$\cdots \to H^q(S^n) \xrightarrow{p^*} H^q(P^n) \xrightarrow{\Psi^*} H^{q+1}(P^n) \xrightarrow{p^*} H^{q+1}(S^n) \to \cdots$$

starting on the left with $0 \to H^0(P^n) \xrightarrow{p^*} H^0(S^n)$ and terminating on the right with $H^n(S^n) \xrightarrow{p^*} H^n(P^n) \to 0$ [note that $H^q(P^n) = 0$ for $q > n$, because P^n is a polyhedron of dimension n]. Because $H^q(S^n) = 0$ for $0 < q < n$, it follows that

$$\Psi^*\colon H^q(P^n) \to H^{q+1}(P^n)$$

is an epimorphism for $0 \leq q < n - 1$ and is a monomorphism for $0 < q \leq n - 1$. Because P^n and S^n are connected for $n \geq 1$, $p^* H^0(P^n) = H^0(S^n)$, which implies that $\Psi^*: H^0(P^n) \to H^1(P^n)$ is also a monomorphism. Therefore $H^q(P^n) \neq 0$ for $0 \leq q \leq n$, and because $\rho^* H^n(S^n) = H^n(P^n)$ and $H^n(S^n) \approx \mathbf{Z}_2$, it follows that ρ^* is a monomorphism and that $\Psi^*: H^{n-1}(P^n) \to H^n(P^n)$ is also an epimorphism.

We have shown that for $0 \leq q \leq n - 1$

$$\Psi^*: H^q(P^n) \approx H^{q+1}(P^n)$$

Then $w_n = \Psi^*(1)$ is the nonzero element of $H^1(P^n)$, and by equation 5.7.15, $\Psi^*(w_n{}^q) = w_n{}^{q+1}$. Therefore, for $1 \leq q \leq n$, $w_n{}^q$ is the nonzero element of $H^q(P^n)$. ∎

By corollary 3.8.9, $P_n(\mathbf{C})$ and $P_n(\mathbf{Q})$ are simply connected. It follows from corollary 5.7.20 that the Hopf bundles $S^{2n+1} \to P_n(\mathbf{C})$ with fiber S^1 and $S^{4n+3} \to P_n(\mathbf{Q})$ with fiber S^3 are orientable over any R. Let $x_n \in H^2(P_n(\mathbf{C});R)$ and $y_n \in H^4(P_n(\mathbf{Q});R)$ be the characteristic classses of these Hopf bundles (based on some orientation class of each bundle). An argument analogous to that of theorem 4, using the Thom-Gysin sequences of the Hopf bundles, establishes the following result.

5 THEOREM *For $n \geq 1$, $H^*(P_n(\mathbf{C});R)$ is a truncated polynomial algebra over R generated by x_n of degree 2 and height $n + 1$, and $H^*(P_n(\mathbf{Q});R)$ is a truncated polynomial algebra over R generated by y_n of degree 4 and height $n + 1$.* ∎

6 COROLLARY *Let $n > m \geq 1$ and let i: $P^m \subset P^n$ be a linear imbedding. Then for $q \leq m$*

$$i^*: H^q(P^n;\mathbf{Z}_2) \approx H^q(P^m,\mathbf{Z}_2)$$

PROOF The hypothesis that i is a linear imbedding implies that the 0-sphere bundle over P^m induced by i from the double covering $S^n \to P^n$ is the double covering $S^m \to P^m$. By the naturality of the characteristic class, $i^* w_n = w_m$. The result now follows from theorem 4 and the fact that $i^*(w_n{}^q) = (i^* w_n)^q$. ∎

7 COROLLARY *Let $n > m \geq 1$ and let f: $P^n \to P^m$ be a map. There exists a map f': $P^n \to S^m$ such that $p \circ f' = f$, where p: $S^m \to P^m$ is the double covering.*

PROOF By the lifting theorem 2.4.5, it suffices to prove $f_\#(\pi(P^n)) = 0$. If $m = 1$, this follows from the fact that $\pi(P^n) = \mathbf{Z}_2$ and $\pi(P^1) = \mathbf{Z}$. Assume that $m > 1$ and observe that because $H^1(P^n)$ has just the two elements 0 and w_n, either $f^*(w_m) = 0$ or $f^*(w_m) = w_n$. Because f^* is an algebra homomorphism, the latter is impossible [since $0 \neq w_n{}^{m+1}$ and $f^*(w_m{}^{m+1}) = 0$]. Therefore $f^*(w_m) = 0$.

We know that $\pi(P^n) = \mathbf{Z}_2$, and a generator for this group is the homotopy class of the linear inclusion map i: $P^1 \subset P^n$. Because $f^*(w_m) = 0$, it follows that $i^*f^*(w_m) = 0$. If j: $P^1 \subset P^m$ is the linear inclusion map, by corollary 6, $j^*(w_m) \neq 0$. Since $(f \circ i)^*(w_m) \neq j^*(w_m)$, $f \circ i$ is not homotopic to j. Since $\pi(P^m) = \mathbf{Z}_2$, $f \circ i$ is null homotopic. Hence $f_\#[i] = [f \circ i] = 0$, and so $f_\#(\pi(P^n)) = 0$ in this case also. ∎

8 COROLLARY *For $n > m \geq 1$ there is no continuous map $g: S^n \to S^m$ such that $g(-x) = -g(x)$ for all $x \in S^n$.*

PROOF If there were such a map, it would define a map $f: P^n \to P^m$ such that the following square (where p and p' are the double coverings) is commutative

$$\begin{array}{ccc} S^n & \xrightarrow{g} & S^m \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ P^n & \xrightarrow{f} & P^m \end{array}$$

By corollary 7, f can be lifted to a map $f': P^n \to S^m$. Then

$$pf'p' = fp' = pg$$

Therefore $f'p'$ and g are liftings of the same map. For any $x \in S^n$ either $g(x) = f'p'(x)$ or $g(-x) = f'p'(x) = f'p'(-x)$. In any event, $f'p'$ and g must agree at some point of S^n. By the unique-lifting property 2.2.2, $f'p' = g$. This is a contradiction, because for any $x \in S^n$, p' maps x and $-x$ into the same point, while g maps them into separate points. ∎

This last result is equivalent to the *Borsuk-Ulam theorem*, which is next.

9 THEOREM *Given a continuous map $f: S^n \to R^n$ for $n \geq 1$, there exists $x \in S^n$ such that $f(x) = f(-x)$.*

PROOF Assume there is no such x and let $g: S^n \to S^{n-1}$ be the map defined by

$$g(x) = \frac{f(x) - f(-x)}{\|f(x) - f(-x)\|}$$

Then $g(-x) = -g(x)$, which would contradict corollary 8. ∎

Dual to the concept of graded R algebra is that of graded R coalgebra, which is defined by dualizing the concept of product. A *graded R coalgebra* consists of a graded R module $A = \{A^q\}$ and a homomorphism of degree 0

$$d: A \to A \otimes A$$

called the *coproduct* of the coalgebra (so d maps A^q into $\bigoplus_{i+j=q} A^i \otimes A^j$ for all q). The coproduct is said to be *associative* if

$$(d \otimes 1)d = (1 \otimes d)d: A \to A \otimes A \otimes A$$

and is said to be *commutative* if $Td = d$, where $T: A \otimes A \to A \otimes A$ is the homomorphism $T(a \otimes a') = (-1)^{\deg a \deg a'} a' \otimes a$. A *counit* for the coalgebra is a homomorphism $\varepsilon: A \to R$ (where R is regarded as a graded R module consisting of R in degree 0) such that each of the composites

$$\begin{tikzcd} & & R \otimes A \arrow[dr, "\approx"] & \\ A \arrow[r, "d"] & A \otimes A \arrow[ur, "\varepsilon \otimes 1"] \arrow[dr, "1 \otimes \varepsilon"'] & & A \\ & & A \otimes R \arrow[ur, "\approx"'] & \end{tikzcd}$$

is the identity map.

A *Hopf algebra over* R is a graded R algebra B which is also a coalgebra whose coproduct

$$d: B \to B \otimes B$$

is a homomorphism of graded R algebras. A Hopf algebra B is said to be *connected* if B^0 is the free R module generated by a unit element 1 for the algebra and the homomorphism $\varepsilon: B \to R$ defined by $\varepsilon(\alpha 1) = \alpha$ for $\alpha \in R$ is a counit for the coalgebra.

10 EXAMPLE If X is a connected H space whose homology over a field R is of finite type, then the multiplication map $\mu: X \times X \to X$ defines a coproduct

$$d = \mu^*: H^*(X;R) \to H^*(X;R) \otimes H^*(X;R)$$

$H^*(X;R)$ with this coproduct is a connected Hopf algebra of finite type whose product is associative and commutative (the fact that X has a homotopy unit x_0 implies that the map $H^*(X;R) \to H^*(x_0;R) \approx R$ is a counit).

We shall study connected Hopf algebras having an associative and commutative product and describe the algebra structure of those which are of finite type over a field of characteristic 0. The following is the inductive step of the structure theorem toward which we are heading.

11 LEMMA *Let B be a connected Hopf algebra with an associative and commutative product over a field R of characteristic* 0. *Let B' be a connected sub Hopf algebra of B such that B is generated as an algebra by B' and some element $x \in B - B'$. If x has odd degree n, then as a graded algebra $B \approx B' \otimes E_n(x)$ and if x has even degree n, then as a graded algebra $B \approx B' \otimes S_n(x)$.*

PROOF Because B' is a sub Hopf algebra of B, the unit element of B belongs to B'. Since $x \in B - B'$, x has positive degree n. Let A be the ideal in B generated by the elements of positive degree in B', and if $\eta: B \to B/A$ is the projection, let

$$d' = (1 \otimes \eta)d: B \to B \otimes B \to B \otimes (B/A)$$

Then d' is an algebra homomorphism, $d'(\beta) = \beta \otimes 1$ for $\beta \in B'$, and $d'(x) = x \otimes 1 + 1 \otimes \eta(x)$. Note that $x \notin A$, because A consists of finite sums $\sum_{i \geq 0} \beta_i x^i$, where $\beta_i \in B'$ is of positive degree, so $\beta_i x^i$ is of degree larger than n unless $i = 0$. Therefore $\eta(x) \neq 0$ in B/A.

Assume that x is of odd degree. Because B has a commutative product and R has characteristic different from 2, $x^2 = 0$. We show that there is no relation of the form $\beta_0 + \beta_1 x = 0$ with $\beta_0, \beta_1 \in B'$ and $\beta_1 \neq 0$. If there were such a relation, then

$$\begin{aligned} 0 = d'(\beta_0 + \beta_1 x) &= \beta_0 \otimes 1 + (\beta_1 \otimes 1)[x \otimes 1 + 1 \otimes \eta(x)] \\ &= \beta_1 \otimes \eta(x) \end{aligned}$$

Since $\eta(x) \neq 0$, this implies $\beta_1 = 0$, which is a contradiction. Therefore the homomorphism $B' \otimes E_n(x) \to B$ sending $\beta \otimes 1$ to β and $\beta \otimes x$ to βx is an isomorphism of graded algebras.

Assume that x is of even degree. We shall show that there is no relation of the form $\Sigma_{0 \le i \le r} \beta_i x^i = 0$ with $\beta_i \in B'$, $r \geq 1$, and $\beta_r \neq 0$. If there were such a relation, consider one of minimal degree in x. Then

$$\begin{aligned} 0 = d'(\Sigma\, \beta_i x^i) &= \Sigma\, (\beta_i \otimes 1)[x \otimes 1 + 1 \otimes \eta(x)]^i \\ &= (\Sigma\, i\beta_i x^{i-1}) \otimes \eta(x) + \cdots + \beta_r \otimes (\eta(x))^r \end{aligned}$$

The only term on the right in $B \otimes (B/A)^n$ is the term $(\Sigma\, i\beta_i x^{i-1}) \otimes \eta(x)$. It must be 0, and because $\eta(x) \neq 0$, $\Sigma\, i\beta_i x^{i-1} = 0$. If $r > 1$, this is a relation of smaller degree in x (note that $r\beta_r \neq 0$ because R has characteristic 0), and this is a contradiction. If $r = 1$, we get $\beta_1 = 0$, which is also a contradiction. Therefore there is no relation, and the homomorphism $B' \otimes S_n(x) \to B$ sending $\beta \otimes x^q$ to βx^q for $\beta \in B'$ and $q \geq 0$ is an isomorphism of graded algebras. ■

We use this result to establish the following *Leray structure theorem* for Hopf algebras over a field of characteristic 0[1].

12 THEOREM *Let B be a connected Hopf algebra with an associative and commutative product and of finite type over a field R of characteristic 0. As a graded R algebra either $B \approx R$ or B is the tensor product of a countable number of exterior algebras with generators of odd degree and a countable number of polynomial algebras with generators of even degree.*

PROOF Because B is of finite type, there is a countable sequence $1 = x_0, x_1, x_2, \ldots$ of elements of B such that $i < j$ implies that $\deg x_i \leq \deg x_j$ and such that as an algebra B is generated by the set $\{x_j\}_{j \geq 0}$. For $n \geq 0$ let B_n be the subalgebra of B generated by $x_0, x_1, \ldots, x_n$. We can also assume that x_{n+1} does not belong to B_n. Because of the condition that $\deg x_j$ is a nondecreasing function of j, each B_n is a connected sub Hopf algebra of B (that is, d maps B_n into $B_n \otimes B_n$). Since B_{n+1} is generated as an algebra by B_n and x_{n+1}, lemma 11 applies. Since $B_0 \approx R$, $B_1 \approx R \otimes E(x_1)$ or $B_1 \approx R \otimes S(x_1)$. Therefore $B = B_0 \approx R$ or B_1 is either an exterior algebra on an odd-degree generator or a polynomial algebra on an even-degree generator. By induction on n, using lemma 11, each B_{n+1} is a tensor product of the desired form. Since B has finite type, $B \approx \lim_{\to} B_n$, and B has the desired form. ■

[1] A structure theorem valid over a perfect field of arbitrary characteristic can be found in A. Borel, Sur la cohomologie des espaces fibrés principaux et des espaces homogenes de groupes de Lie compacts, *Annals of Mathematics*, vol. 57, pp. 115–207, 1953.

For a connected H space whose homology is finitely generated over a field F no polynomial algebra factors can occur in the above structure theorem, and we obtain the following *Hopf theorem on H spaces.*

13 COROLLARY *Let X be a connected H space whose homology over a field R of characteristic 0 is finitely generated. Then the cohomology algebra of X over R is isomorphic to the cohomology algebra over R of a product of a finite number of odd-dimensional spheres.* ■

In particular, we obtain the following result about spheres that can be H spaces.

14 COROLLARY *No even-dimensional sphere of positive dimension is an H space.* ■

9 THE STEENROD SQUARING OPERATIONS

In the last section the cup product in cohomology was used to prove the Borsuk-Ulam theorem, a geometric result. Any other algebraic structure which can be introduced into cohomology (or homology) and which is functorial can be similarly applied. A particular example of such an additional algebraic structure is a natural transformation from one cohomology functor to another. These natural transformations are called cohomology operations. In this section we introduce the concept of cohomology operation and define the particular set of cohomology operations called the Steenrod squares.

Let p and q be fixed integers and G and G' fixed R modules. A *cohomology operation θ of type* $(p,q;\ G,G')$ is a natural transformation from the functor $H^p(\quad;G)$ to the functor $H^q(\quad;G')$ (both functors being contravariant singular cohomology functors defined on the category of topological pairs). Thus θ assigns to a pair (X,A) a function (which is not assumed to be a homomorphism)

$$\theta_{(X,A)}\colon H^p(X,A;\ G) \to H^q(X,A;\ G')$$

such that if $f\colon (X,A) \to (Y,B)$ is a map, there is a commutative square

$$\begin{array}{ccc} H^p(Y,B;\ G) & \xrightarrow{\theta_{(Y,B)}} & H^q(Y,B;\ G') \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ H^p(X,A;\ G) & \xrightarrow{\theta_{(X,A)}} & H^q(X,A;\ G') \end{array}$$

A *homology operation* is defined similarly, but we shall not discuss homology operations.

Following are some examples.

1 If $\varphi\colon G \to G'$ is a homomorphism, φ_* is a cohomology operation of type $(q,q;\ G,G')$ for every q, where

$$\varphi_*\colon H^q(X,A;\ G) \to H^q(X,A;\ G')$$

is defined as in Sec. 5.4. φ_* is called the *operation induced by the coefficient homomorphism* φ.

2 Given a short exact sequence of R modules $0 \to G' \to G \to G'' \to 0$, the *Bockstein cohomology operation* β^* of type $(q, q+1; G'',G')$ for every q is defined to equal the Bockstein homomorphism

$$\beta^*: H^q(X,A; G'') \to H^{q+1}(X,A; G')$$

corresponding to the coefficient sequence $0 \to G' \to G \to G'' \to 0$ as defined in theorem 5.4.11.

3 For any p and q there is an operation θ_p of type $(q,pq; R,R)$, called the *pth-power operation*, defined by

$$\theta_p(u) = u^p \qquad u \in H^q(X,A; R)$$

An operation θ is said to be *additive* if $\theta_{(X,A)}$ is a homomorphism for every (X,A). The operations in examples 1 and 2 are additive; however, the operation θ_p of example 3 is not additive, in general.

Any cohomology operation provides a necessary condition for a homomorphism between the cohomology modules of two pairs to be the induced homomorphism of some continuous map between the pairs. For example, if θ is of type $(p,q; G,G)$, a necessary condition that a homomorphism

$$\psi: H^*(Y,B; G) \to H^*(X,A; G)$$

be induced by some map $f: (X,A) \to (Y,B)$ is that

$$\psi\theta_{(Y,B)} = \theta_{(X,A)}\psi: H^p(Y,B; G) \to H^q(X,A; G)$$

In these terms the algebraic idea underlying corollaries 5.8.7 and 5.8.8 is that for $n > m \geq 1$ there is no homomorphism

$$\psi: H^*(P^m;\mathbf{Z}_2) \to H^*(P^n;\mathbf{Z}_2)$$

such that ψ sends the nonzero element of $H^1(P^m;\mathbf{Z}_2)$ to the nonzero element of $H^1(P^n;\mathbf{Z}_2)$ and commutes with the $(m+1)$st-power operation θ_{m+1} of type $(1, m+1; \mathbf{Z}_2,\mathbf{Z}_2)$.

We shall now define a sequence of operations Sq^i called the Steenrod squares, each Sq^i being a cohomology operation of type $(q, q+i; \mathbf{Z}_2,\mathbf{Z}_2)$ for every q. These operations include the squaring operation θ_2 and are related to it by "reducing" the value of $\theta_2(u)$ in a certain way. For this reason, the operations Sq^i are also called the reduced squares.

For the remainder of this section we make the assumption that all modules are over $\mathbf{Z}_2$ and all homology and cohomology modules have coefficients $\mathbf{Z}_2$. The *Steenrod squares*, or *reduced squares*, $\{Sq^i\}_{i\geq 0}$ are additive cohomology operations

$$Sq^i: H^q(X,A) \to H^{q+i}(X,A)$$

defined for all q such that

(a) $Sq^0 = 1$.
(b) If $\deg u = q$, then $Sq^q u = u \smile u$.
(c) If $q > \deg u$, then $Sq^q u = 0$.
(d) If $u \in H^*(X,A)$ and $v \in H^*(Y,B)$ and $\{X \times B, A \times Y\}$ is an excisive couple in $X \times Y$, the following *Cartan formula* is valid:

$$Sq^k(u \times v) = \sum_{i+j=k} Sq^i u \times Sq^j v$$

The above properties characterize the cohomology operations Sq^i. We shall not prove the uniqueness[1], but shall content ourselves with their construction. First we establish a formula equivalent to the Cartan formula.

4 LEMMA *If $u, v \in H^*(X,A)$, then*

$$Sq^k(u \smile v) = \sum_{i+j=k} Sq^i u \smile Sq^j v$$

PROOF Since $u \smile v = d^*(u \times v)$, where $d\colon (X,A) \to (X,A) \times (X,A)$ is the diagonal map, this follows from the Cartan formula and functorial properties of Sq^i. ■

For any chain complex C let $T\colon C \otimes C \to C \otimes C$ be the chain map interchanging the factors [$T(c_1 \otimes c_2) = c_2 \otimes c_1$ is a chain map over $\mathbf{Z}_2$].

5 LEMMA *There exists a sequence $\{D_j\}_{j \geq 0}$ of functorial homomorphisms $D_j\colon \Delta(X) \to \Delta(X) \otimes \Delta(X)$ of degree j such that*

(a) *D_0 is a chain map commuting with augmentation.*
(b) *For $j > 0$, $\partial D_j + D_j \partial + D_{j-1} + TD_{j-1} = 0$.*

If $\{D_j\}$ and $\{D'_j\}$ are two such sequences, there exists a sequence $\{E_j\}_{j \geq 0}$ of functorial homomorphisms $E_j\colon \Delta(X) \to \Delta(X) \otimes \Delta(X)$ of degree j such that

(c) *$E_0 = 0$.*
(d) *For $j \geq 0$, $\partial E_{j+1} + E_{j+1}\partial + E_j + TE_j + D_j + D'_j = 0$.*

PROOF We use the method of acyclic models. Let R be the group ring of $\mathbf{Z}_2$ over the field $\mathbf{Z}_2$. We regard R as the quotient ring of the polynomial ring $\mathbf{Z}_2(t)$ modulo the ideal generated by the polynomial $t^2 + 1 = 0$. Thus the elements of R have the form $a + bt$, where a and $b \in \mathbf{Z}_2$.

Let $\mathbf{Z}_2$ be regarded as a trivial R module (that is, the element t of R induces the identity map of $\mathbf{Z}_2$) and let C be the free resolution of $\mathbf{Z}_2$ over R in which C_q is free with one generator d_q for all $q \geq 0$ and which has boundary operator $\partial(d_q) = (1 + t)d_{q-1}$ for $q \geq 1$ and augmentation $\varepsilon(d_0) = 1$. The functor which assigns to a space X the chain complex $\Delta(X) \underset{\mathbf{Z}_2}{\otimes} C$ is augmented and free over R with models $\{\Delta_q\}_{q \geq 0}$ and basis $\{\xi_q \otimes d_j\}$. We regard

[1] For a proof see N. Steenrod and D. Epstein, Cohomology operations, *Annals of Mathematics Studies No.* 50, Princeton University Press, Princeton, N.J., 1962.

$\Delta(X) \underset{\mathbf{Z}_2}{\otimes} \Delta(X)$ as a chain complex over R, with t acting on $\Delta(X) \otimes \Delta(X)$ in the same way T does. Then $\Delta(X) \otimes \Delta(X)$ is augmented and acyclic, with models $\{\Delta_q\}_{q\geq 0}$. It follows from theorem 4.3.3 (which is valid for chain complexes over R) that there exist natural chain maps τ: $\Delta(X) \otimes C \to \Delta(X) \otimes \Delta(X)$ preserving augmentation, and any two are naturally chain homotopic.

A map τ: $\Delta(X) \otimes C \to \Delta(X) \otimes \Delta(X)$ of degree 0 corresponds bijectively to a sequence of maps

$$D_j\colon \Delta(X) \to \Delta(X) \otimes \Delta(X) \qquad j \geq 0$$

of degree j such that $D_j(c) = \tau(c \otimes d_j)$. Then τ is a chain map preserving augmentation if and only if $\{D_j\}$ satisfies (*a*) and (*b*). Thus there exist families $\{D_j\}$ satisfying (*a*) and (*b*), and any such family corresponds to some τ.

Similarly, a map H: $\Delta(X) \otimes C \to \Delta(X) \otimes \Delta(X)$ of degree 1 corresponds bijectively to a sequence of maps

$$E_j\colon \Delta(X) \to \Delta(X) \otimes \Delta(X) \qquad j \geq 0$$

of degree j such that $E_0 = 0$ and $E_j(c) = H(c \otimes d_{j-1})$ for $j \geq 1$. Then H is a chain homotopy from τ to τ' if and only if $\{E_j\}$ satisfies (*c*) and (*d*) for the sequences $\{D_j\}$ and $\{D'_j\}$ corresponding to τ and τ', respectively. Thus, if $\{D_j\}$ and $\{D'_j\}$ are two sequences satisfying (*a*) and (*b*), there is a sequence $\{E_j\}$ satisfying (*c*) and (*d*). ∎

Given a sequence $\{D_j\}_{j\geq 0}$ as in lemma 5, we define homomorphisms

$$D_j^*\colon \operatorname{Hom}(\Delta(X) \otimes \Delta(X), \mathbf{Z}_2) \to \operatorname{Hom}(\Delta(X), \mathbf{Z}_2)$$

of degree $-j$ by $(D_j^* f)(\sigma) = f(D_j\sigma)$ for $\sigma \in \Delta_q(X)$ and $f \in \operatorname{Hom}(\Delta(X) \otimes \Delta(X), \mathbf{Z}_2)$. If $c^* \in \operatorname{Hom}(\Delta_q(X), \mathbf{Z}_2)$ is a q-cochain of $\Delta(X)$, then

$$c^* \otimes c^* \in \operatorname{Hom}(\Delta(X) \otimes \Delta(X), \mathbf{Z}_2),$$

and we define a $(q + i)$-cochain $\mathrm{Sq}^i c^* \in \operatorname{Hom}(\Delta(X), \mathbf{Z}_2)$ by

$$Sq^i c^* = \begin{cases} 0 & i > q \\ D_{q-i}^*(c^* \otimes c^*) & i \leq q \end{cases}$$

Let us now establish some properties of these cochain maps. It will be convenient to understand $D_j = 0$ for $j < 0$. Then lemma 5*b* holds for all j.

6 *If c^* is zero on $\Delta(A)$ for some $A \subset X$, then $Sq^i c^*$ is zero on $\Delta(A)$.*

PROOF This follows from the naturality of $\{D_j\}$, and hence of $\{Sq^i\}$. ∎

7 *If $\delta c^* = 0$, then $\delta(Sq^i c^*) = 0$.*

PROOF This is trivial if $i > q$. If $i \leq q$, we have

$$\begin{aligned} \delta(Sq^i c^*)(\sigma) &= D_{q-i}^*(c^* \otimes c^*)(\partial\sigma) = (c^* \otimes c^*)(D_{q-i}\partial\sigma) \\ &= (c^* \otimes c^*)(\partial D_{q-i}\sigma) + (c^* \otimes c^*)(D_{q-i-1}\sigma + TD_{q-i-1}\sigma) \\ &= (c^* \otimes c^*)(\partial D_{q-i}\sigma) \end{aligned}$$

the last equality because $(c^* \otimes c^*)(Tc) = (c^* \otimes c^*)c$ for any $c \in \Delta(X) \otimes \Delta(X)$. Then we have

$$(c^* \otimes c^*)(\partial D_{q-i}\sigma) = \delta(c^* \otimes c^*)(D_{q-i}\sigma) = 0$$

because $\delta c^* = 0$. ■

8 *If $c^* = \delta \bar{c}^*$, then $Sq^i c^* = \delta[D^*_{q-i}(\bar{c}^* \otimes c^*) + D^*_{q-i-1}(\bar{c}^* \otimes \bar{c}^*)]$.*

PROOF If $i > q$, both sides are zero. If $i \leq q$, we have

$$\begin{aligned}(Sq^i c^*)(\sigma) &= D^*_{q-i}(\delta\bar{c}^* \otimes \delta\bar{c}^*)(\sigma) = \delta(\bar{c}^* \otimes \delta\bar{c}^*)(D_{q-i}(\sigma))\\ &= (\bar{c}^* \otimes \delta\bar{c}^*)(D_{q-i}\partial\sigma + D_{q-i-1}\sigma + TD_{q-i-1}\sigma)\\ &= D^*_{q-i}(\bar{c}^* \otimes c^*)(\partial\sigma) + \delta(\bar{c}^* \otimes \bar{c}^*)(D_{q-i-1}\sigma)\end{aligned}$$

the last equality because

$$(\bar{c}^* \otimes \delta\bar{c}^*)(D_{q-i-1}\sigma + TD_{q-i-1}\sigma) = (\bar{c}^* \otimes \delta\bar{c}^* + \delta\bar{c}^* \otimes \bar{c}^*)(D_{q-i-1}\sigma)$$

We also have

$$\begin{aligned}\delta(\bar{c}^* \otimes \bar{c}^*)(D_{q-i-1}\sigma) &= (\bar{c}^* \otimes \bar{c}^*)(D_{q-i-1}\partial\sigma + D_{q-i-2}\sigma + TD_{q-i-2}\sigma)\\ &= D^*_{q-i-1}(\bar{c}^* \otimes \bar{c}^*)(\partial\sigma)\end{aligned}$$

The result follows by substituting this into the right-hand side of the other equation. ■

9 *If c_1^* and c_2^* are cocycles, then*

$$Sq^i(c_1^* + c_2^*) = Sq^i c_1^* + Sq^i c_2^* + \delta D^*_{q-i+1}(c_1^* \otimes c_2^*)$$

PROOF If $i > q$, both sides are zero. If $i \leq q$, we have

$$\begin{aligned}Sq^i(c_1^* + c_2^*)(\sigma) &= [(c_1^* + c_2^*) \otimes (c_1^* + c_2^*)](D_{q-i}\sigma)\\ &= (c_1^* \otimes c_1^* + c_2^* \otimes c_2^*)(D_{q-i}\sigma) + (c_1^* \otimes c_2^*)(D_{q-i}\sigma + TD_{q-i}\sigma)\\ &= (Sq^i c_1^* + Sq^i c_2^*)(\sigma) + (c_1^* \otimes c_2^*)(D_{q-i+1}\partial\sigma + \partial D_{q-i+1}\sigma)\\ &= [Sq^i c_1^* + Sq^i c_2^* + \delta D^*_{q-i+1}(c_1^* \otimes c_2^*)](\sigma)\end{aligned}$$

the last equality because $\delta(c_1^* \otimes c_2^*) = 0$. ■

It follows that there is a well-defined functorial homomorphism

$$Sq^i: H^q(X,A) \to H^{q+i}(X,A)$$

defined by $Sq^i\{c^*\} = \{Sq^i c^*\}$. If $\{D'_j\}$ is another system satisfying lemma 5*a* and 5*b*, and Sq'^i is defined using this system, let $\{E_j\}$ satisfy 5*c* and 5*d*. If c^* is a q-cocycle of $\Delta(X)/\Delta(A)$, then

$$(c^* \otimes c^*)(D_{q-i}\sigma + D'_{q-i}\sigma + E_{q+1-i}\partial\sigma) = 0$$

Therefore

$$Sq^i c^* + Sq'^i c^* + \delta E^*_{q+1-i}(c^* \otimes c^*) = 0$$

showing that $Sq^i\{c^*\} = Sq'^i\{c^*\}$. Hence Sq^i is uniquely defined independent of the particular choice of $\{D_j\}$. We shall now verify that these cohomology operations $\{Sq^i\}$ satisfy the axioms characterizing the Steenrod squares.

10 THEOREM *The additive cohomology operations $\{Sq^i\}$ defined above satisfy conditions (a) to (d), inclusive, on page 271.*

PROOF Let $C(\Delta^q)$ denote the oriented chain complex of the simplex. Over $\mathbf{Z}_2$ there is a unique orientation for each simplex, and $C(\Delta^q)$ is isomorphic to the subcomplex of $\Delta(\Delta^q)$ generated by the singular simplexes which are the faces of Δ^q. We regard $C(\Delta^q)$ as imbedded in $\Delta(\Delta^q)$ in this way. $\tilde{C}(\Delta^q)$ is acyclic, and if $\lambda\colon \Delta^p \to \Delta^q$ is a p-face of Δ^q, then $\Delta(\lambda)(C(\Delta^p)) \subset C(\Delta^q)$. It follows that a sequence $\{D_j\}$ can be found satisfying lemma 5*a* and 5*b* such that $D_j(\xi_q) \in C(\Delta^q) \otimes C(\Delta^q)$ for all q and j. For such a sequence, $D_j(\xi_q) = 0$ if $j > q$ (because $[C(\Delta^q) \otimes C(\Delta^q)]_s = 0$ if $s > 2q$), whence $D_j(\sigma) = 0$ for any $\sigma \in \Delta_q(X)$ with $q < j$.

We now shall prove $D_q(\xi_q) = \xi_q \otimes \xi_q$ for all q by induction on q. If $q = 0$, then $D_0(\xi_0)$ must have nonzero augmentation, by lemma 5*a*. The only element of $C(\Delta^0) \otimes C(\Delta^0)$ with nonzero augmentation is $\xi_0 \otimes \xi_0$. Therefore $D_0(\xi_0) = \xi_0 \otimes \xi_0$. Assume that $q > 0$ and $D_{q-1}(\xi_{q-1}) = \xi_{q-1} \otimes \xi_{q-1}$. Either $D_q(\xi_q) = \xi_q \otimes \xi_q$ or $D_q(\xi_q) = 0$. In the latter case, by lemma 5*b*, we have [because $D_q(\partial\xi_q) = 0$]

$$D_{q-1}(\xi_q) + TD_{q-1}(\xi_q) = 0$$

From this it follows that $D_{q-1}(\xi_q) = \sum a_i(\xi_q \otimes \xi_q^{(i)} + \xi_q^{(i)} \otimes \xi_q)$, where $a_i = 0$ or $a_i = 1$. This is a contradiction, because

$$D_{q-2}(\xi_q) + TD_{q-2}(\xi_q) = \partial D_{q-1}(\xi_q) + D_{q-1}(\partial\xi_q)$$

and $\xi_q^{(i)} \otimes \xi_q^{(i)}$ has a coefficient of $2a_i + 1 = 1$ on the right and a coefficient of 0 on the left.

Therefore, with this choice of $\{D_j\}$ we have $D_q(\sigma) = \sigma \otimes \sigma$ if σ has degree q. Then

$$(Sq^0 c^*)(\sigma) = (c^* \otimes c^*)(D_q(\sigma)) = [c^*(\sigma)]^2$$

Because $a^2 = a$ for $a \in \mathbf{Z}_2$, we see that $Sq^0 c^* = c^*$, and so $Sq^0 = 1$, showing that condition (*a*) is satisfied.

By definition, D_0 is a chain approximation to the diagonal. Therefore $\{D_0^*(c^* \otimes c^*)\} = \{c^*\} \smile \{c^*\}$ for any cocycle c^*, and so $Sq^q u = u \smile u$ if $\deg u = q$. Hence condition (*b*) is satisfied. From the definition of Sq^i condition (*c*) is trivially satisfied.

It merely remains to verify the Cartan formula. Let $\{D_j\}$ be a system satisfying lemma 5*a* and 5*b* and let $\{D_j^X\}$ be the collection of homomorphisms for $\Delta(X)$. On the category of pairs of topological spaces X and Y the system $\{D_k^{X\times Y}\}$ and the system $\{\bar{T} \sum_{i+j=k} T^k D_i^X \otimes D_j^Y\}$, where

$$\bar{T}\colon [\Delta(X) \otimes \Delta(X)] \otimes [\Delta(Y) \otimes \Delta(Y)] \to [\Delta(X) \otimes \Delta(Y)] \otimes [\Delta(X) \otimes \Delta(Y)]$$

interchanges the second and third factors, both satisfy lemma 5*a* and 5*b*.

Then a system $\{E_k^{X\times Y}\}$ satisfying 5*c* and 5*d* with respect to them can be defined by the method of acyclic models. Therefore the system

$$\{\bar{T} \sum_{i+j=k} T^k D_i^X \otimes D_j^Y\}$$

can be used to define $Sq^k(u \times v)$ for $u \in H^*(X,A)$ and $v \in H^*(Y,B)$. Let c_1^* be a p-cochain of X, c_2^* a q-cochain of Y, σ_1 a singular p'-simplex of X with $p \le p' \le 2p$, and σ_2 a singular q'-simplex of Y with $q \le q' \le 2q$, where $p' + q' = p + q + k$. Then

$$\begin{aligned}
Sq^k(c_1^* \otimes c_2^*)(\sigma_1 \otimes \sigma_2) \\
&= [(c_1^* \otimes c_2^*) \otimes (c_1^* \otimes c_2^*)](D^{X\times Y}_{p+q-k}(\sigma_1 \otimes \sigma_2)) \\
&= [(c_1^* \otimes c_1^*) \otimes (c_2^* \otimes c_2^*)](\sum_{i+j=p+q-k} T^{p+q-k} D_i^X\sigma_1 \otimes D_j^Y\sigma_2) \\
&= [(c_1^* \otimes c_1^*)(D^X_{2p-p'}\sigma_1)]\,[(c_2^* \otimes c_2^*)(D^Y_{2q-q'}\sigma_2)] \\
&= (Sq^{p'-p}c_1^* \otimes Sq^{q'-q}c_2^*)(\sigma_1 \otimes \sigma_2)
\end{aligned}$$

Letting σ_1 and σ_2 vary, we see that $Sq^k(c_1^* \otimes c_2^*) = \sum_{i+j=k} Sq^i c_1^* \otimes Sq^j c_2^*$. Passing to cohomology and using the natural homomorphism

$$H^*(X,A) \otimes H^*(Y,B) \to H^*([\Delta(X)/\Delta(A)] \otimes [\Delta(Y)/\Delta(B)]) \approx H^*((X,A) \times (Y,B))$$

sending the tensor product to the cross product, we obtain

$$Sq^k(u \times v) = \sum_{i+j=k} Sq^i u \times Sq^j v$$

showing that condition (*d*) is satisfied. ∎

11 EXAMPLE Observe that, by condition (*b*) on page 271 and theorem 5.8.5,

$$Sq^2\colon H^2(P_2(\mathbf{C})) \to H^4(P_2(\mathbf{C}))$$

is nontrivial. If $u \in H^2(P_2(\mathbf{C}))$ is such that $Sq^2u \ne 0$ and $v \in H^1(I,\dot{I})$ is the nontrivial element, it follows from condition (*d*) that

$$Sq^2(u \times v) = Sq^2u \times v$$

and $Sq^2\colon H^3(P_2(\mathbf{C}) \times (I,\dot{I})) \to H^5(P_2(\mathbf{C}) \times (I,\dot{I}))$ is nontrivial. Let X be the unreduced suspension of $P_2(\mathbf{C})$ obtained from $P_2(\mathbf{C}) \times I$ by identifying $P_2(\mathbf{C}) \times 0$ to one point x_0 and $P_2(\mathbf{C}) \times 1$ to another point x_1. There is then a continuous map

$$f\colon P_2(\mathbf{C}) \times (I,\dot{I}) \to (X, x_0 \cup x_1)$$

inducing an isomorphism

$$f^*\colon H^q(X, x_0 \cup x_1) \approx H^q(P_2(\mathbf{C}) \times (I,\dot{I}))$$

for all q. Therefore $Sq^2\colon H^3(X) \to H^5(X)$ is nontrivial. Let Y be the one-point union of S^3 and S^5. An easy computation shows that X and Y have isomorphic homology and cohomology for any coefficient group, and even isomorphic cup and cap products. However, because $Sq^2\colon H^3(X) \to H^5(X)$ is nontrivial and $Sq^2\colon H^3(Y) \to H^5(Y)$ is trivial, X and Y are not of the same homotopy type.

Further applications of the Steenrod squares will be given in the next chapter and in Chap. 8.

It is obvious that cohomology operations of the same type can be added and that the sum is again a cohomology operation of the same type. Given cohomology operations θ of type $(p,q;\ G,G')$ and θ' of type $(q,r;\ G',G'')$, their composite $\theta'\theta$ (of natural transformations) is a cohomology operation of type $(p,r;\ G,G'')$. In this way the Steenrod squares can be added and multiplied, and they generate an algebra of cohomology operations called the *modulo 2 Steenrod algebra.*

In this algebra the following *Adem relations*[1] hold:

$$Sq^i Sq^j = \sum_{0 \le k \le [i/2]} \binom{j-k-1}{i-2k} Sq^{i+j-k} Sq^k \qquad 0 < i < 2j$$

where $[i/2]$ denotes as usual the largest integer $\le i/2$ and the binomial coefficient $\binom{j-k-1}{i-2k}$ is reduced modulo 2. Using these relations, it is easily shown that the algebra of cohomology operations generated by Sq^i, where i is a power of 2, contains all the Steenrod squares. This implies that the only spheres that can be H spaces have dimension $2^n - 1$ for some n. By using deeper properties of the algebra of cohomology operations Adams[2] has shown that the only spheres that can be H spaces are the spheres S^0, S^1, S^3, and S^7. Each of these is, in fact, an H space, with multiplication defined to be the multiplication of the reals, complex numbers, quaternions, or Cayley numbers, respectively, of norm 1.

EXERCISES

A DISSECTIONS

Let C be a graded module over R. A *filtration* (*increasing*) of C is a sequence $\{F_sC\}$ of graded submodules of C such that $F_sC \subset F_{s+1}C$ for all s. It is said to be *bounded below* if for any t there is $s(t)$ such that $F_{s(t)}C_t = 0$, and it is *convergent above* if $\cup F_sC = C$.

1 If $\{F_sC\}$ is a filtration of a chain complex C by subcomplexes, there is an increasing filtration of $H_*(C)$ defined by $F_sH_*(C) = \text{im}\,[H_*(F_sC) \to H_*(C)]$. If the original filtration on C is bounded below or convergent above, prove that the same is true of the induced filtration on $H_*(C)$.

An increasing filtration $\{F_sC\}$ of a chain complex C by subcomplexes is called a *dissection* if it is bounded below, convergent above, and if

$$H_q(F_{s+1}C,F_sC) = 0 \qquad q \ne s+1$$

[1] See J. Adem, The iteration of the Steenrod squares in algebraic topology, *Proceedings of the National Academy of Sciences, USA*, vol. 38, pp. 720–726, 1952, or H. Cartan, Sur l'iteration des operations de Steenrod, *Commentarii Mathematici Helvetici*, vol. 29, pp. 40–58, 1955.

[2] See J. F. Adams, On the non-existence of elements of Hopf invariant one, *Annals of Mathematics*, vol. 72, pp. 20–104, 1960.

Given a dissection $\{F_sC\}$ of a chain complex C, the sequence

$$\cdots \to H_{q+1}(F_{q+1}C,F_qC) \xrightarrow{\partial} H_q(F_qC,F_{q-1}C) \xrightarrow{\partial} H_{q-1}(F_{q-1}C,F_{q-2}C) \to \cdots$$

is a chain complex $\bar{C}$, called the *chain complex associated to the dissection.*

2 If $\bar{C}$ is the chain complex associated to a dissection of C, prove that $H_*(\bar{C}) \approx H_*(C)$.

3 Let $\{F_sC\}$ be a dissection of a free chain complex C by free subcomplexes such that $F_{s+1}C/F_sC$ is free for all s. If $\bar{C}$ is the chain complex associated to the dissection, prove that $\bar{C}$ and C have isomorphic homology and cohomology for all coefficient modules. [*Hint:* The freeness hypotheses ensure that the universal-coefficient theorems hold for both homology and cohomology. Then $\{F_sC \otimes G\}$ is a dissection of $C \otimes G$ whose associated chain complex is isomorphic to $\bar{C} \otimes G$. Dual considerations apply to $\{\operatorname{Hom}(F_sC,G)\}$ and $\operatorname{Hom}(C,G)$.]

A *block dissection* of a chain complex C is a collection of subcomplexes $\{E_j{}^q\}$, called *blocks,* where q varies over the set of integers and for each q, j varies over a set J_q, such that if F_sC is the subcomplex of C generated by $\{E_j{}^q\}_{q\le s}$ and if $\dot{E}_i{}^q = E_j{}^q \cap F_{s-1}C$, then

$$\begin{aligned} E_j{}^q \cap E_k{}^q &\subset F_{q-1}C && j \ne k \\ E_j{}^q &= 0 && q \text{ sufficiently small} \\ \cup\, F_sC &= C \\ H_i(E_j{}^q,\dot{E}_j{}^q) &\approx \begin{cases} 0 & i \ne q \\ R & i = q \end{cases} \end{aligned}$$

4 If $\{E_j{}^q\}$ is a block dissection of a chain complex C, prove that the corresponding collection $\{F_sC\}$ is a dissection of C whose associated chain complex $\bar{C}$ is free with generators for $\bar{C}_q$ in one-to-one correspondence with the set J_q.

A *block dissection* of a simplicial complex K is a collection of subcomplexes $\{K_j{}^q\}$, where q varies over the set of integers and for each q, j varies over some indexing set J_q, such that if $F_sK = \cup_{j\le s} K_j{}^q$ and $\dot{K}_j{}^q = F_{s-1}K \cap K_j{}^q$, then

$$\begin{aligned} K_j{}^q \cap K_k{}^q &\subset F_{q-1}K && j \ne k \\ K_j{}^q &= 0 && q \text{ sufficiently small} \\ \cup\, F_sK &= K \\ H_i(K_j{}^q,\dot{K}_j{}^q) &\approx \begin{cases} 0 & i \ne q \\ Z & i = q \end{cases} \end{aligned}$$

5 If $\{K_j{}^q\}$ is a block dissection of K, prove that $\{C(K_j{}^q)\}$ is a block dissection of the chain complex $C(K)$ by free subcomplexes. If $\bar{C}$ is the chain complex associated to the dissection, prove that $\bar{C}$ and $C(K)$ have isomorphic homology and cohomology with any coefficient group.

B HOMOLOGY MANIFOLDS

A *homology n-manifold* is a locally compact Hausdorff space X such that for all $x \in X$, $H_q(X, X - x) = 0$ for $q \ne n$ and either $H_n(X, X - x) = 0$ or $H_n(X, X - x) \approx \mathbf{Z}$. Furthermore, if the *boundary* $\dot{X}$ of X is defined to be the subset

$$\dot{X} = \{x \in X \mid H_n(X, X - x) = 0\}$$

then we also assume that $X - \dot{X}$ is a nonempty connected set. If $\dot{X} = \varnothing$, X is said to be *without boundary.*

1 If X is a homology n-manifold and Y is a homology m-manifold, prove that $X \times Y$ is a homology $(n + m)$-manifold whose boundary equals $\dot{X} \times Y \cup X \times \dot{Y}$.

2 Prove that if a polyhedron is a homology n-manifold, its boundary is a subpolyhedron.

3 If K is a simplicial complex triangulating a homology n-manifold X, prove that K is an n-dimensional pseudomanifold and $\dot{K}$ triangulates $\dot{X}$. (A polyhedral homology n-manifold is said to be *orientable* or *nonorientable*, according to whether any triangulation of it is orientable or nonorientable as a pseudomanifold.)

4 Let $(K,\dot{K})$ be a simplicial pair triangulating a polyhedral homology n-manifold $(X,\dot{X})$ and let L be the subcomplex of the barycentric subdivision K' consisting of all simplexes disjoint from $\dot{K}'$. If s^q is a q-simplex of $K - \dot{K}$, let $E^{n-q}(s^q)$ be the subcomplex of L generated by the star of the barycenter $b(s^q)$. Prove that $\{E^{n-q}(s^q)\}_{s^q \in K-\dot{K}}$ is a block dissection of L and that if $\bar{C}$ is the chain complex associated to this block dissection, then $\bar{C}$ has homology and cohomology isomorphic to that of $X - \dot{X}$. (*Hint:* let st $s^q = s^q * B(s^q)$, where $B(s^q)$ is a subcomplex of K. Then $E^{n-q}(s^q) = b(s^q) * [B(s^q)]'$ and $\dot{E}^{n-q}(s^q) = [B(s^q)]'$. Also note that $|L|$ is a strong deformation retract of $|K| - |\dot{K}|$.)

5 *Lefschetz duality theorem.* Let $(K,\dot{K})$ be a simplicial pair triangulating a compact homology n-manifold $(X,\dot{X})$ and assume that $z \in H_n(K,\dot{K})$ is an orientation of K. For each q-simplex s^q of $K - \dot{K}$ let $z(s^q) \in H_n(K, K - \text{st } s^q)$ be the image of z, and assume an orientation σ^q of s^q chosen once and for all. Then $z(s^q) = \sigma^q * \bar{z}(\sigma^q)$, where $\bar{z}(\sigma^q) \in H_{n-q-1}(B(s^q))$. Define $z'(\sigma^q) \in H_{n-q}(E^{n-q}(s^q),\dot{E}^{n-q}(s^q))$ to correspond to $\bar{z}(\sigma^q)$ under the isomorphisms

$$H_{n-q-1}(B(s^q)) \approx H_{n-q-1}(\dot{E}^{n-q}(s^q)) \approx H_{n-q}(E^{n-q}(s^q),\dot{E}^{n-q}(s^q))$$

Let φ: Hom $(C_q(K,\dot{K}), G) \to \bar{C}_{n-q} \otimes G$ be the homomorphism defined by

$$\varphi(u) = \sum_{\sigma^q} z'(\sigma^q) \otimes u(\sigma^q) \qquad u \in \text{Hom}\,(C_q(K,\dot{K}), G)$$

Prove that φ is an isomorphism and that it commutes up to sign with the respective coboundary and boundary operators. Deduce isomorphisms

$$H^q(X,\dot{X}; G) \approx H_{n-q}(X - \dot{X}; G) \qquad \text{and} \qquad H_q(X,\dot{X}; G) \approx H^{n-q}(X - \dot{X}; G)$$

C PROPERTIES OF THE TORSION PRODUCT AND EXT

In this group of exercises all modules will be over a principal ideal domain R.

1 Prove that the torsion product is associative.

2 If A, B, and C are modules, prove that

$$A \otimes (B * C) \oplus A * (B \otimes C)$$

is symmetric in A, B, and C.

3 Given a module A and a short exact sequence of modules

$$0 \to B' \to B \to B'' \to 0$$

prove there is an exact sequence

$$0 \to \text{Hom}\,(A,B') \to \text{Hom}\,(A,B) \to \text{Hom}\,(A,B'') \to \\ \text{Ext}\,(A,B') \to \text{Ext}\,(A,B) \to \text{Ext}\,(A,B'') \to 0$$

4 Given a short exact sequence of modules

$$0 \to A' \to A \to A'' \to 0$$

and given a module B, prove there is an exact sequence

$$0 \to \operatorname{Hom}(A'',B) \to \operatorname{Hom}(A,B) \to \operatorname{Hom}(A',B) \to \operatorname{Ext}(A'',B) \to \operatorname{Ext}(A,B) \to \operatorname{Ext}(A',B) \to 0$$

If $C = \{C_i\}$ and $C^* = \{C^j\}$ are graded modules, there is a graded module $\operatorname{Hom}(C,C^*) = \{\operatorname{Hom}^q(C,C^*)\}$, where $\operatorname{Hom}^q(C,C^*) = \times_{i+j=q} \operatorname{Hom}(C_i,C^j)$ [thus an element of $\operatorname{Hom}^q(C,C^*)$ is an indexed family $\{\varphi_i\colon C_i \to C^{q-i}\}_i$]. Similarly, there is a graded module $\operatorname{Ext}(C,C^*) = \{\operatorname{Ext}^q(C,C^*)\}$, where $\operatorname{Ext}^q(C,C^*) = \times_{i+j=q} \operatorname{Ext}(C_i,C^j)$.

5 If C is a chain complex and C^* is a cochain complex, prove that $\operatorname{Hom}(C,C^*)$ is a cochain complex, with

$$(\delta\varphi)_{i,j} = \varphi_{i-1,j} \circ \partial_i + (-1)^i \delta^{j-1} \circ \varphi_{i,j-1} \qquad \varphi = \{\varphi_{i,j}\} \in \operatorname{Hom}^q(C,C^*)$$

and that $\operatorname{Ext}(C,C^*)$ is a cochain complex with

$$(\delta\psi)_{i,j} = \operatorname{Ext}(\partial_i,1)(\psi_{i-1,j}) + (-1)^i \operatorname{Ext}(1,\delta^{j-1})(\psi_{i,j-1}) \qquad \psi = \{\psi_{i,j}\} \in \operatorname{Ext}^q(C,C^*)$$

6 If C is a chain complex and C^* is a cochain complex such that $\operatorname{Ext}(C,C^*)$ is acyclic, prove that there is a split short exact sequence

$$0 \to \operatorname{Ext}^{q-1}(H_*(C),H^*(C^*)) \to H^q(\operatorname{Hom}(C,C^*)) \to \operatorname{Hom}^q(H_*(C),H^*(C^*)) \to 0$$

7 If C and C' are chain complexes and C^* is a cochain complex, prove that the exponential correspondence is an isomorphism

$$\operatorname{Hom}(C, \operatorname{Hom}(C',C^*)) \approx \operatorname{Hom}(C \otimes C', C^*)$$

8 Let (X,A) and (Y,B) be topological pairs such that $\{X \times B, A \times Y\}$ is an excisive couple in $X \times Y$. For any module G prove that there is a split short exact sequence

$$0 \to \operatorname{Ext}^{q-1}(H_*,H^*) \to H^q((X,A) \times (Y,B); G) \to \operatorname{Hom}^q(H_*,H^*) \to 0$$

where $H_* = H_*(X,A; R)$ and $H^* = H^*(Y,B; G)$.

D CATEGORY

A topological space X is said to have *category* $\leq n$, denoted as cat $X \leq n$, if X is the union of n closed sets, each deformable to a point in X.

1 If X is a connected polyhedron of dimension n, prove that cat $X \leq n + 1$.

2 If X is any space, prove that cat $(SX) \leq 2$.

3 If cat $X \leq n$, prove that all n-fold cup products of positive-dimensional cohomology classes of X vanish.

4 Prove that cat $P^n = n + 1$ and cat $(P^{n_1} \times \cdots \times P^{n_k}) = n_1 + \cdots + n_k + 1$.

E HOMOLOGY OF FIBER BUNDLES

1 Let $p\colon E \to B$ be a fiber-bundle pair, with total pair $(E,\dot E)$ and fiber pair $(F,\dot F)$, such that $H_*(F,\dot F) = 0$. Prove that $H_*(E,\dot E) = 0$.

2 If $p\colon E \to B$ is a fiber-bundle pair over a path-connected base space B, prove that a homomorphism $\theta\colon H^*(F,\dot F; R) \to H^*(E,\dot E; R)$ is a cohomology extension of the fiber if and only if for some $b \in B$ the composite

$$H^*(F,\dot F; R) \xrightarrow{\theta} H^*(E,\dot E; R) \to H^*(E_b,\dot E_b; R)$$

is an isomorphism.

3 Let $p\colon E \to B$ be a fiber-bundle pair over a path-connected base space. If for some $b \in B$ the pair $(E_b,\dot E_b)$ is a weak retract of $(E,\dot E)$, prove there exists a cohomology extension of the fiber.

4 Prove that a q-sphere bundle ξ with base space B is orientable over R if and only if for every map $\alpha: S^1 \to B$ the bundle $\alpha^*(\xi)$ is orientable over R.

5 Prove that a q-sphere bundle ξ is orientable over $\mathbf{Z}$ if and only if there is an element $U \in H^{q+1}(E_\xi,\dot{E}_\xi;\ \mathbf{Z}_4)$ whose image in $H^{q+1}(E_\xi,\dot{E}_\xi;\ \mathbf{Z}_2)$ is the unique orientation class of ξ over $\mathbf{Z}_2$. (*Hint:* Show that there is such an element U if and only if for every closed path ω in the base space, $h[\omega]^*$ is the identity map of $H^{q+1}(E_{\omega(1)},\dot{E}_{\omega(1)};\ \mathbf{Z}_4)$, and this, in turn, is equivalent to the condition that $h[\omega]^*$ is the identity map of $H^{q+1}(E_{\omega(1)},\dot{E}_{\omega(1)};\ \mathbf{Z})$.)

6 Let ξ be a q-sphere bundle with base space B and with orientation class $U_\xi \in H^{q+1}(E_\xi,\dot{E}_\xi;\ R)$ and let $\Omega_\xi \in H^{q+1}(B;R)$ be the corresponding characteristic class. Prove that $\Phi_\xi^*(\Omega_\xi) = U_\xi \smile U_\xi$.

7 Prove that the characteristic class Ω_ξ of an even-dimensional sphere bundle ξ oriented over $\mathbf{Z}$ has order 2.

8 Let ξ be a sphere bundle oriented over R, with base space B. If ξ has a section in $\dot{E}_\xi$, (that is, if the map $\dot{p}_\xi$: $\dot{E}_\xi \to B$ has a right inverse), prove that its characteristic class $\Omega_\xi = 0$. [*Hint:* Any two sections $B \to E_\xi$ are homotopic in E_ξ. Since E_ξ is the mapping cylinder of $\dot{p}_\xi$: $\dot{E}_\xi \to B$, there is an inclusion map k: $B \subset E_\xi$ which is a section. There is a section in $\dot{E}_\xi$ if and only if k is homotopic to a map $B \to \dot{E}_\xi$, in which case the composite

$$H^{q+1}(E_\xi,\dot{E}_\xi;\ R) \xrightarrow{i^*} H^{q+1}(E_\xi;R) \xrightarrow{p^{*-1}} H^{q+1}(B;R)$$

is trivial, because $p^{*-1} = k^*$.]

F HOPF ALGEBRAS

1 Prove that the tensor product of connected Hopf algebras is a connected Hopf algebra.

2 If B is a connected Hopf algebra of finite type over a field R, prove that $B^* = \operatorname{Hom}(B;R)$ is a connected Hopf algebra over R whose product and coproduct are dual, respectively, to the coproduct and product of B.

3 Let B be a connected Hopf algebra over a field of characteristic $p \neq 0$ and assume that B has an associative and commutative product and is generated as an algebra by a single element x of positive degree. Prove that if deg x is odd and $p \neq 2$, then $B = E(x)$, and if deg x is even or $p = 2$, then either $B = S_{\deg x}(x)$ or $B = T_{\deg x,h}(x)$, where $h = p^k$ for some $k \geq 1$.

4 Let B be a connected Hopf algebra of finite type over a field of finite characteristic $p \neq 0$ and assume that B has an associative and commutative product. If the pth power of every element of positive degree of B is 0, prove that B is the tensor product of exterior algebras (with generators of odd degree if $p \neq 2$) and truncated polynomial algebras of height p (with generators of even degree if $p \neq 2$).

G THE BOCKSTEIN HOMOMORPHISM

1 Show that the Bockstein homomorphism in homology (or cohomology) anticommutes with the boundary homomorphism (or coboundary homomorphism) of a pair.

For any prime p let β_p be the Bockstein homomorphism in either homology or cohomology for the short exact sequence

$$0 \to \mathbf{Z}_p \to \mathbf{Z}_{p^2} \to \mathbf{Z}_p \to 0$$

Let $\tilde{\beta}_p$ be the Bockstein homomorphism for the short exact sequence

$$0 \to \mathbf{Z} \xrightarrow{\lambda_p} \mathbf{Z} \xrightarrow{\mu_p} \mathbf{Z}_p \to 0$$

where $\lambda_p(n) = pn$ and μ_p is reduction modulo p.

2 Prove that $\beta_p = (\mu_p)_* \circ \bar{\beta}_p$.

3 Prove that $\beta_p \circ \beta_p = 0$.

4 Prove that $\beta_p(u \smile v) = \beta_p(u) \smile v + (-1)^{\deg u} u \smile \beta_p(v)$.

5 Prove that $Sq^{2i+1} = \beta_2 \circ Sq^{2i}$ for $i \geq 0$. [*Hint:* Show that there exist functorial homomorphisms $\{D_j\}_{j\geq 0}$, with D_j of degree j from the integral singular chain complex $\Delta(X)$ to $\Delta(X) \otimes \Delta(X)$, such that D_0 is a chain map commuting with augmentation and

$$\begin{aligned} \partial D_{2j-1} + D_{2j-1}\partial &= D_{2j} - TD_{2j} \qquad & j \geq 0 \\ \partial D_{2j} - D_{2j}\partial &= D_{2j-1} + TD_{2j-1} \qquad & j > 0 \end{aligned}$$

where $T(\sigma_1 \otimes \sigma_2) = (-1)^{\deg \sigma_1 \deg \sigma_2} \sigma_2 \otimes \sigma_1$.]

6 Let ξ be a q-sphere bundle and let $U_\xi \in H^{q+1}(E_\xi,\dot{E}_\xi; \mathbf{Z}_2)$ be its unique orientation over $\mathbf{Z}_2$. Prove that ξ is orientable over $\mathbf{Z}$ if and only if $\beta_2(U_\xi) = 0$.

H STIEFEL-WHITNEY CHARACTERISTIC CLASSES

Let ξ be a q-sphere bundle, with base space B, and let $U_\xi \in H^{q+1}(E_\xi,\dot{E}_\xi; \mathbf{Z}_2)$ be its orientation class over $\mathbf{Z}_2$. The ith *Stiefel-Whitney characteristic class* $w_i(\xi) \in H^i(B;\mathbf{Z}_2)$ for $i \geq 0$ is defined by

$$\Phi^*_\xi(w_i(\xi)) = Sq^i(U_\xi)$$

1 Let $f: B' \to B$ be continuous. Prove that $f^*(w_i(\xi)) = w_i(f^*\xi)$.

2 If ξ is a product bundle, prove that $w_i(\xi) = 0$ for $i > 0$.

3 Prove the following:

(*a*) $w_0(\xi)$ is the unit class of $H^0(B;\mathbf{Z}_2)$.
(*b*) $\beta_2(w_{2i}(\xi)) = w_{2i+1}(\xi) + w_1(\xi) \smile w_{2i}(\xi)$ for $i \geq 0$.
(*c*) If ξ is a q-sphere bundle, then $w_i(\xi) = 0$ for $i > q + 1$, and $w_{q+1}(\xi)$ is the characteristic class of ξ over $\mathbf{Z}_2$.
(*d*) ξ is orientable over $\mathbf{Z}$ if and only if $w_1(\xi) = 0$.

If ξ is a q-sphere bundle over B and ξ' is a q'-sphere bundle over B', their *cross product* $\xi \times \xi'$ is a $(q + q' + 1)$-sphere bundle with $E_{\xi\times\xi'} = E_\xi \times E_{\xi'}$, $\dot{E}_{\xi\times\xi'} = E_\xi \times \dot{E}_{\xi'} \cup \dot{E}_\xi \times E_{\xi'}$ and $p_{\xi\times\xi'} = p_\xi \times p_{\xi'}$.

4 If $U_\xi \in H^{q+1}(E_\xi,\dot{E}_\xi; \mathbf{Z}_2)$ and $U_{\xi'} \in H^{q'+1}(E_{\xi'},\dot{E}_{\xi'}; \mathbf{Z}_2)$ are respective orientation classes, prove that

$$U_\xi \times U_{\xi'} \in H^{q+q'+2}(E_{\xi\times\xi'},\dot{E}_{\xi\times\xi'}; \mathbf{Z}_2)$$

is the orientation class of $\xi \times \xi'$.

5 Prove that $w_k(\xi \times \xi') = \Sigma_{i+j=k}\, w_i(\xi) \times w_j(\xi')$.

If ξ and ξ' are sphere bundles with the same base space B, their *Whitney sum* $\xi \oplus \xi'$ is the sphere bundle over B induced from $\xi \times \xi'$ by the diagonal map $B \to B \times B$.

6 *Whitney duality theorem.* Prove that

$$w_k(\xi \oplus \xi') = \sum_{i+j=k} w_i(\xi) \smile w_j(\xi')$$

I HOMOLOGY WITH LOCAL COEFFICIENTS

If $\sigma: \Delta^q \to X$ is a singular q-simplex of X, with $q \geq 1$, let ω_σ be the path in X obtained by composing the linear path in Δ^q from v_0 to v_1 with σ. Given a local system Γ of

R modules on X, define $\Delta_q(X;\Gamma)$ to be the R module of finitely nonzero formal sums $\sum \alpha_\sigma \sigma$ in which σ varies over the set of singular q-simplexes of X and $\alpha_\sigma \in \Gamma(\sigma(v_0))$ is zero except for a finite set of σ. For $q > 0$ define a homomorphism ∂: $\Delta_q(X;\Gamma) \to \Delta_{q-1}(X;\Gamma)$ by

$$\partial(\alpha\sigma) = \sum_{0<i\leq q} (-1)^i \alpha\sigma^{(i)} + \Gamma(\omega_\sigma)(\alpha)\sigma^{(0)}$$

1 Prove that $\Delta(X;\Gamma) = \{\Delta_q(X;\Gamma), \partial\}$ is a chain complex which is free (or torsion free) if Γ is a local system of free (or torsion free) R modules, and if $A \subset X$, show that $\Delta(A;\, \Gamma \mid A)$ is a subcomplex of $\Delta(X;\Gamma)$.

The *homology of* (X,A) *with local coefficients* Γ, denoted by $H_*(X,A;\, \Gamma)$, is defined to be the graded homology module of $\Delta(X,A;\, \Gamma) = \Delta(X;\Gamma)/\Delta(A;\, \Gamma \mid A)$.

2 For a fixed ring R let $\mathcal{C}$ be the category whose objects are topological pairs (X,A), together with local systems Γ of R modules on X, and whose morphisms from (X,A) and Γ to (Y,B) and Γ' are continuous maps f: $(X,A) \to (Y,B)$, together with indexed families of homomorphisms $\{f_x\colon \Gamma(x) \to \Gamma'(f(x))\}_{x\in X}$ such that $f_{\omega(0)} \circ \Gamma(\omega) = \Gamma'(f \circ \omega) \circ f_{\omega(1)}$ for any path ω in X. Prove that $H_*(X,A;\, \Gamma)$ is a covariant functor from $\mathcal{C}$ to the category of graded R modules.

3 *Exactness.* Given $A \subset B \subset X$ and a local system Γ of R modules on X, prove that there is an exact sequence

$$\cdots \to H_q(B,A;\, \Gamma \mid B) \to H_q(X,A;\, \Gamma) \to H_q(X,B;\, \Gamma) \to H_{q-1}(B,A;\, \Gamma \mid B) \to \cdots$$

4 *Excision.* Let X_1 and X_2 be subsets of a space X such that $X_1 \cup X_2 = \operatorname{int} X_1 \cup \operatorname{int} X_2$. For any local system Γ of R modules on X prove that the excision map j_1 from $(X_1, X_1 \cap X_2)$ and $\Gamma \mid X_1$ to $(X_1 \cup X_2, X_2)$ and $\Gamma \mid (X_1 \cup X_2)$ induces an isomorphism

$$j_{1*}\colon H_*(X_1, X_1 \cap X_2;\, \Gamma \mid X_1) \approx H_*(X_1 \cup X_2, X_2;\, \Gamma \mid (X_1 \cup X_2))$$

5 Two morphisms f and g in $\mathcal{C}$ from (X,A) and Γ to (Y,B) and Γ' are said to be *homotopic* in $\mathcal{C}$ if there is a homotopy F: $(X,A) \times I \to (Y,B)$ from f to g and an indexed family of homomorphisms $\{F_{(x,t)}\colon \Gamma(x) \to \Gamma'(F(x,t))\}_{(x,t)\in X\times I}$ such that $F_{(x,0)} = f_x$ and $F_{(x,1)} = g_x$. Prove that homotopy is an equivalence relation in the set of morphisms from (X,A) and Γ to (Y,B) and Γ' and that the composites of homotopic morphisms are homotopic (so that the homotopy category of $\mathcal{C}$ can be defined).

6 *Homotopy.* If f and g are morphisms from (X,A) and Γ to (Y,B) and Γ' and f is homotopic to g in $\mathcal{C}$, prove that $f_* = g_*\colon H_*(X,A;\, \Gamma) \to H_*(Y,B;\, \Gamma')$.

7 If Γ and Γ' are local systems of R modules on X, there is a local system $\Gamma \otimes \Gamma'$ on X with $(\Gamma \otimes \Gamma')(x) = \Gamma(x) \otimes \Gamma'(x)$ and $(\Gamma \otimes \Gamma')(\omega) = \Gamma(\omega) \otimes \Gamma'(\omega)$. In case Γ' is the constant local system equal to G, then prove that

$$\Delta(X,A;\, \Gamma \otimes G) \approx \Delta(X,A;\, \Gamma) \otimes G$$

Deduce a universal-coefficient formula for homology with local coefficients.

8 If Γ and Γ' are local systems of R modules on X and Y, respectively, let $\Gamma \times \Gamma' = p^*(\Gamma) \otimes p'^*(\Gamma')$ be the local system on $X \times Y$, where $p^*(\Gamma)$ and $p'^*(\Gamma')$ are induced from Γ and Γ', respectively, by the projections p: $X \times Y \to X$ and p': $X \times Y \to Y$. Prove that there is a natural chain equivalence of $\Delta(X;\Gamma) \otimes \Delta(Y;\Gamma')$ with $\Delta(X \times Y;\, \Gamma \times \Gamma')$. Deduce a Künneth formula for homology with local coefficients.

J COHOMOLOGY WITH LOCAL COEFFICIENTS

If Γ is a local system of R modules on X, define $\Delta^q(X;\Gamma)$ to be the module of functions φ assigning to every singular q-simplex σ of X an element $\varphi(\sigma) \in \Gamma(\sigma(v_0))$. Define a homomorphism δ: $\Delta^q(X;\Gamma) \to \Delta^{q+1}(X;\Gamma)$ by

$$(\delta\varphi)(\sigma) = \sum_{0<i\leq q+1} (-1)^i\varphi(\sigma^{(i)}) + \Gamma(\omega_\sigma^{-1})(\varphi(\sigma^{(0)}))$$

1 Prove that $\Delta^*(X;\Gamma) = \{\Delta^q(X;\Gamma), \delta\}$ is a cochain complex and that if $A \subset X$, the restriction map $\Delta^*(X;\Gamma) \to \Delta^*(A;\ \Gamma \mid A)$ is an epimorphism.

The *cohomology of* (X,A) *with local coefficients* Γ, denoted by $H^*(X,A;\ \Gamma)$, is defined to be the graded cohomology module of

$$\Delta^*(X,A;\ \Gamma) = \ker\,[\Delta^*(X;\Gamma) \to \Delta^*(A;\ \Gamma \mid A)]$$

2 For a fixed ring R let $\mathcal{C}'$ be the category whose objects are topological pairs (X,A), together with local systems Γ of R modules on X, and whose morphisms from (X,A) and Γ to (Y,B) and Γ' are continuous maps $f\colon (X,A) \to (Y,B)$, together with indexed families of homomorphisms $\{f^x\colon \Gamma'(f(x)) \to \Gamma(x)\}_{x\in X}$ such that $\Gamma(\omega) \circ f^{\omega(1)} = f^{\omega(0)} \circ \Gamma'(f \circ \omega)$ for any path ω in X. Prove that $H^*(X,A;\ \Gamma)$ is a contravariant functor from $\mathcal{C}'$ to the category of graded R modules.

3 Prove that the cohomology with local coefficients has exactness, excision, and homotopy properties analogous to those of the homology with local coefficients.

4 If Γ is a local system of R modules on X and G is an R module, there is a local system Hom (Γ,G) of R modules on X which assigns to $x \in X$ the module Hom $(\Gamma(x),G)$. Prove that

$$\Delta^*(X,A;\ \mathrm{Hom}\,(\Gamma,G)) \approx \mathrm{Hom}\,(\Delta(X,A;\ \Gamma),\ G)$$

Deduce a universal-coefficient formula for cohomology with local coefficients.

Let ξ be a q-sphere bundle with base space B and let Γ_ξ be the local system on B such that $\Gamma_\xi(b) = H_{q+1}(E_b,\dot{E}_b)$. Let $p_\xi^*(\Gamma_\xi)$ be the local system on E_ξ induced from Γ_ξ by $p_\xi\colon E_\xi \to B$. A *Thom class* of ξ is an element $U_\xi \in H^{q+1}(E_\xi,\dot{E}_\xi;\ p_\xi^*(\Gamma_\xi))$ such that for every $b \in B$ the element

$$U_\xi \mid (E_b,\dot{E}_b) \in H^{q+1}(E_b,\dot{E}_b;\ p_\xi^*(\Gamma_\xi) \mid E_b) = H^{q+1}(E_b,\dot{E}_b;\ H_{q+1}(E_b,\dot{E}_b))$$

corresponds to the identity map of $H_{q+1}(E_b,\dot{E}_b)$ under the universal-coefficient isomorphism

$$H^{q+1}(E_b,\dot{E}_b;\ H_{q+1}(E_b,\dot{E}_b)) \approx \mathrm{Hom}\,(H_{q+1}(E_b,\dot{E}_b),\ H_{q+1}(E_b,\dot{E}_b))$$

5 Prove that every q-sphere bundle has a unique Thom class. (*Hint:* Prove the result first for a product bundle, and then use Mayer-Vietoris sequences to extend the result to arbitrary bundles.)

6 Let ξ be a q-sphere bundle with a base space B and let U_ξ be its Thom class. If Γ is any local system of abelian groups on X, prove that the homomorphism

$$\Phi_\xi\colon H_n(E_\xi,\dot{E}_\xi;\ p^*(\Gamma)) \to H_{n-q-1}(B;\ \Gamma_\xi \otimes \Gamma)$$

such that $\Phi_\xi(z) = p_*(U_\xi \frown z)$, where $U_\xi \frown z$ is an element of $H_{n-q-1}(E;\ p^*(\Gamma_\xi \otimes \Gamma))$, is an isomorphism. If B is compact, prove that the homomorphism

$$\Phi_\xi^*\colon H^r(B;\Gamma) \to H^{r+q+1}(E_\xi,\dot{E}_\xi;\ p^*(\Gamma \otimes \Gamma_\xi))$$

such that $\Phi_\xi^*(v) = p^*(v) \smile U_\xi$ is an isomorphism.

CHAPTER SIX

GENERAL COHOMOLOGY THEORY AND DUALITY

IN THIS CHAPTER WE CONTINUE THE STUDY OF HOMOLOGY AND COHOMOLOGY functors, with particular emphasis on the homological properties of topological manifolds. For this important class of spaces we shall establish the duality theorem equating the cohomology of a compact pair in an orientable manifold with the homology, in complementary dimensions, of the complementary pair.

The cohomology which enters in the duality theorem is the direct limit of the singular cohomology of neighborhoods of the pair, with the family of neighborhoods directed downward by inclusion. For the case of a closed pair in a manifold, the resulting direct limit depends only on the pair itself. In fact, it is isomorphic to the Alexander cohomology of the pair, Alexander cohomology being another cohomology theory distinct from the singular cohomology.

Thus we are led to consider Alexander cohomology. We define it and prove that it is a cohomology theory in the sense that it satisfies the axioms of cohomology theory. We also establish the special properties of tautness and continuity possessed by this theory and not generally valid for singular cohomology. For deeper properties of the Alexander theory we introduce the cohomology of a space with coefficients in a presheaf. The definition of this cohomology involves a Čech construction, using nerves of open coverings. We use general properties of this cohomology to prove that for paracompact spaces the Alexander and Čech cohomologies are isomorphic, and with this result establish universal-coefficient formulas for the Alexander cohomology of compact pairs and for the Alexander cohomology with compact supports of locally compact pairs.

The cohomology of presheaves is also applied to compare the singular and Alexander cohomology theories, and we prove that they are isomorphic for manifolds. Another application of the cohomology of presheaves is in the proof of the Vietoris-Begle mapping theorem. The final topic is a discussion of homological properties of one manifold imbedded in another.

In Sec. 6.1 we define the slant product as a pairing from the cohomology of a product space and the homology of one of its factors to the cohomology of the other factor. This furnishes the map that is the isomorphism in the duality theorem for manifolds, and the duality theorem itself is proved in Sec. 6.2. In Sec. 6.3 we consider various formulations of orientability for manifolds.

The Alexander cohomology theory is defined in Secs. 6.4 and 6.5, and the axioms of cohomology theory are verified for it. Section 6.6 contains a proof of the tautness property for Alexander cohomology, that the Alexander cohomology of a closed pair in a paracompact space is isomorphic to the direct limit of the Alexander cohomology of its neighborhoods. We deduce the continuity property of Alexander cohomology and show that the continuity property characterizes Alexander cohomology on compact pairs. We also define the Alexander cohomology with compact supports.

Sections 6.7, 6.8, and 6.9 develop the theory of the cohomology of spaces with coefficients in a presheaf and illustrate its application to the Alexander theory. In this way we equate the Alexander and singular cohomology for paracompact spaces that are homologically locally connected in all dimensions.

Section 6.10 contains definitions of the characteristic classes of a manifold and the normal characteristic classes of one manifold imbedded in another. These are related in the Whitney duality theorem, which is a useful tool for establishing non-imbeddability results.

1 THE SLANT PRODUCT

We are ready now to introduce a new product which pairs cohomology of a product space and homology of one of the factors to the cohomology of the other factor. This product will be used in the next section to prove the duality theorem for topological manifolds. In this section we shall establish some of its properties. We shall also introduce new cohomology modules of a pair (A,B) in a space X which appear to depend on the imbedding of (A,B) in X. These will be used in the proof of the duality theorem in the next section. Later in the chapter, we shall introduce the Alexander cohomology modules and prove that these are isomorphic to the abovementioned ones in all relevant cases.

Given chain complexes C and C' over R and a cochain

$$c^* \in \operatorname{Hom}((C \otimes C')_n, G)$$

and chain $c' \in C'_q \otimes G'$, their *slant product* $c^*/c' \in \operatorname{Hom}(C_{n-q}, G \otimes G')$ is the $(n-q)$-cochain such that if $c' = \sum_i c'_i \otimes g'_i$ with $c'_i \in C'_q$ and $g'_i \in G'$, then

$$\langle c^*/c', c\rangle = \sum_i \langle c^*, c \otimes c'_i\rangle \otimes g'_i \qquad c \in C_{n-q}$$

It is easily verified that

$$\delta(c^*/c') = [(\delta c^*)/c'] + (-1)^{n-q} c^*/\partial c'$$

Therefore the slant product of a cocycle and a cycle is a cocycle, and if the cocycle is a coboundary or the cycle is a boundary, the slant product is a coboundary. Hence there is a slant product of $H^n(C \otimes C'; G)$ and $H_q(C';G')$ to $H^{n-q}(C; G \otimes G')$ such that $\{c^*\}/\{c'\} = \{c^*/c'\}$ for $\{c^*\} \in H^n(C \otimes C'; G)$ and $\{c'\} \in H_q(C';G')$.

For topological pairs (X,A) and (Y,B) let

$$\tau\colon [\Delta(X)/\Delta(A)] \otimes [\Delta(Y)/\Delta(B)] \to [\Delta(X \times Y)]/[\Delta(X \times B \cup A \times Y)]$$

be a functorial chain map given by the Eilenberg-Zilber theorem. For $u \in H^n((X,A) \times (Y,B); G)$ and $z \in H_q(Y,B; G')$, their *slant product*

$$u/z \in H^{n-q}(X,A; G \otimes G')$$

is defined to equal the slant product $(\tau^* u)/z$. The following properties of this slant product are easy consequences of the definitions.

1 *Given $f\colon (X,A) \to (X',A')$, $g\colon (Y,B) \to (Y',B')$, $u \in H^n((X',A') \times (Y',B'); G)$, and $z \in H_q(Y,B; G')$, then, in $H^{n-q}(X,A; G \otimes G')$,*

$$[(f \times g)^* u]/z = f^*(u/g_* z) \quad \blacksquare$$

2 *Given $u \in H^p(X,A; G)$, $v \in H^q(Y,B; G')$, and $z \in H_q(Y,B; G'')$, if $\{X \times B, A \times Y\}$ is an excisive couple in $X \times Y$, then, in $H^p(X,A; G \otimes G' \otimes G'')$,*

$$(u \times v)/z = \mu(u \otimes \langle v,z\rangle) \quad \blacksquare$$

3 *Let $\{(X_1,A_1), (X_2,A_2)\}$ and $\{(Y_1,B_1), (Y_2,B_2)\}$ be excisive couples in X and Y, respectively. Given*

$$u \in H^n((X_1 \cup X_2) \times (Y_1 \cup Y_2), X_1 \times B_1 \cup X_2 \times B_2 \cup A_1 \times Y_1 \cup A_2 \times Y_2; G)$$

and

$$z \in H_q(Y_1 \cup Y_2, B_1 \cup B_2; G')$$

then, in $H^{n-q+1}(X_1 \cup X_2, A_1 \cup A_2; G \otimes G')$,

$$\begin{aligned}&[u \mid (X_1 \cup X_2, A_1 \cup A_2) \times (Y_1 \cap Y_2, B_1 \cap B_2)]/\partial_* z \\ &\qquad = (-1)^{n-q-1}\delta^*([u \mid (X_1 \cap X_2, A_1 \cap A_2) \times (Y_1 \cup Y_2, B_1 \cup B_2)]/z) \quad \blacksquare\end{aligned}$$

The following formulas express relations between the slant product and the cup and cap products. We sketch proofs in which the Alexander-Whitney diagonal approximation $\sigma \to \sum_{i+j=\deg \sigma} {}_i\sigma \otimes \sigma_j$ is used in $\Delta(X)$ and its tensor product with itself

$$\sigma \otimes \sigma' \to \sum_{i,j} (-1)^{j(p-i)} ({}_i\sigma \otimes {}_j\sigma') \otimes (\sigma_{p-i} \otimes \sigma'_{q-j}) \qquad \deg \sigma = p, \deg \sigma' = q$$

is used in $\Delta(X) \otimes \Delta(Y)$.

4 *Given* $v \in H^p(X,A;\, G)$, $u \in H^n((X,A') \times (Y,B);\, G')$, *and* $z \in H_q(Y,B;\, G'')$, *then, in* $H^{p+n-q}(X,\, A \cup A';\; G \otimes G' \otimes G'')$,

$$v \smile (u/z) = [(v \times 1) \smile u]/z$$

PROOF Let c_1^* be a p-cochain of $\Delta(X)$, c_2^* an n-cochain of $\Delta(X) \otimes \Delta(Y)$, and $\sigma' \in \Delta_q(Y)$. It suffices to prove that

$$c_1^* \smile (c_2^*/\sigma') = [(c_1^* \otimes 1) \smile c_2^*]/\sigma'$$

If $\sigma \in \Delta_{p+n-q}(X)$, then

$$\begin{aligned} \langle c_1^* \smile (c_2^*/\sigma'), \sigma\rangle &= \langle c_1^*, {}_p\sigma\rangle \otimes \langle c_2^*/\sigma', \sigma_{n-q}\rangle \\ &= \langle c_1^*, {}_p\sigma\rangle \otimes \langle c_2^*, \sigma_{n-q} \otimes \sigma'\rangle \\ &= \langle c_1^* \otimes 1, {}_p\sigma \otimes {}_0\sigma'\rangle \otimes \langle c_2^*, \sigma_{n-q} \otimes \sigma'\rangle \\ &= \langle (c_1^* \otimes 1) \smile c_2^*, \sigma \otimes \sigma'\rangle = \langle [(c_1^* \otimes 1) \smile c_2^*]/\sigma', \sigma\rangle \quad \blacksquare \end{aligned}$$

5 *If* $u \in H^n((X,A) \times (Y,B);\, G)$, $v \in H^p(Y,B';\, G')$, *and* $z \in H_q(Y,B \cup B';\, G'')$, *then, in* $H^{n-(q-p)}(X,A;\, G \otimes G' \otimes G'')$,

$$u/(v \frown z) = [u \smile (1 \times v)]/z$$

PROOF Let c_1^* be an n-cochain of $\Delta(X) \otimes \Delta(Y)$, c_2^* be a p-cochain of $\Delta(Y)$, and $\sigma' \in \Delta_q(Y)$. It suffices to prove that

$$c_1^*/(c_2^* \frown \sigma') = [c_1^* \smile (1 \otimes c_2^*)]/\sigma'$$

If $\sigma \in \Delta_{n-(q-p)}(X)$, then

$$\begin{aligned} \langle c_1^*/(c_2^* \frown \sigma'), \sigma\rangle &= \langle c_1^*, \sigma \otimes (c_2^* \frown \sigma')\rangle \\ &= \langle c_1^*, (1 \otimes c_2^*) \frown (\sigma \otimes \sigma')\rangle \\ &= \langle c_1^* \smile (1 \otimes c_2^*), \sigma \otimes \sigma'\rangle \\ &= \langle [c_1^* \smile (1 \otimes c_2^*)]/\sigma', \sigma\rangle \quad \blacksquare \end{aligned}$$

6 *Given* $u \in H^n((X,A) \times (Y,B);\, G)$, $w \in H_r(X,A;\, G')$, *and* $z \in H_q(Y,B;\, G'')$, *let* $p\colon X \times Y \to X$ *be the projection to the first factor and let*

$$T\colon G \otimes G'' \otimes G' \to G \otimes G' \otimes G''$$

interchange the last two factors. Then, in $H_{r-(n-q)}(X;\, G \otimes G' \otimes G'')$,

$$p_*(u \frown (w \times z)) = T_*[(u/z) \frown w]$$

PROOF Let c^* be an n-cochain of $\Delta(X) \otimes \Delta(Y)$, $\sigma \in \Delta_r(X)$, and $\sigma' \in \Delta_q(Y)$.

Then

$$\begin{aligned}\Delta(p)(c^* \frown (\sigma \otimes \sigma')) &= \Delta(p)[\sum_{i+j=n} (-1)^{i(q-j)}({}_{r-i}\sigma \otimes {}_{q-j}\sigma') \otimes \langle c^*, \sigma_i \otimes \sigma'_j\rangle] \\ &= {}_{r-(n-q)}\sigma \otimes \langle c^*, \sigma_{n-q} \otimes \sigma'\rangle \\ &= {}_{r-(n-q)}\sigma \otimes \langle c^*/\sigma', \sigma_{n-q}\rangle \\ &= (c^*/\sigma') \frown \sigma \quad \blacksquare\end{aligned}$$

For a topological space X let $\delta(X)$ be the *diagonal* of X defined by $\delta(X) = \{(x,x') \in X \times X \mid x = x'\}$. Given $u \in H^n(X \times X, X \times X - \delta(X); R)$ and a pair (A,B) in X, define

$$\gamma_u\colon H_q(X - B, X - A; G) \to H^{n-q}(A,B; G)$$

by $\gamma_u(z) = [u \mid (A,B) \times (X - B, X - A)]/z$ (with $R \otimes G$ identified with G). If $i\colon (A,B) \subset (A',B')$ and $j\colon (X - B', X - A') \subset (X - B, X - A)$, it follows from property 1 that there is a commutative diagram (all coefficients G)

$$\begin{array}{ccc} H_q(X - B', X - A') & \xrightarrow{\gamma_u} & H^{n-q}(A',B') \\ {\scriptstyle j_*}\downarrow & & \downarrow{\scriptstyle i^*} \\ H_q(X - B, X - A) & \xrightarrow{\gamma_u} & H^{n-q}(A,B) \end{array}$$

Thus γ_u is a natural transformation from $H_q(X - B, X - A)$ to $H^{n-q}(A,B)$ on the category of pairs of subspaces and inclusion maps in X. It follows from property 3 that γ_u commutes up to sign with the connecting homomorphisms of relative Mayer-Vietoris sequences.

For a pair (A,B) in a topological space X we define a *neighborhood* (U,V) *of* (A,B) to be a pair in X such that U is a neighborhood of A and V is a neighborhood of B. The family of all neighborhoods of (A,B) in X is directed downward by inclusion. Hence

$$\{H^q(U,V; G) \mid (U,V) \text{ a neighborhood of } (A,B)\}$$

is a direct system, and we define

$$\bar{H}^q(A,B; G) = \lim_{\to} \{H^q(U,V; G)\}$$

where (U,V) varies over neighborhoods of (A,B) [or over the cofinal family of open neighborhoods of (A,B)]. The restriction maps $H^q(U,V; G) \to H^q(A,B; G)$ define a natural homomorphism

$$i\colon \bar{H}^q(A,B; G) \to H^q(A,B; G)$$

The pair (A,B) is said to be *tautly imbedded* in X, or to be a *taut pair in X* (with respect to singular cohomology), if i is an isomorphism for all q and G. The definition of tautness can be formulated for any cohomology theory (or any contravariant functor). We shall see examples later of a subspace taut with respect to one cohomology theory but not with respect to another.

Following are some examples.

7 If (A,B) is an open pair, or, more generally, if it has arbitrarily small neighborhoods which are homotopy equivalent to (A,B), then (A,B) is a taut pair in X.

8 Let $A' = \{(x,y) \in \mathbf{R}^2 \mid x > 0, y = \sin 1/x\}$, let $A'' = \{(x,y) \in \mathbf{R}^2 \mid x = 0, |y| \le 1\}$, and let $A = A' \cup A'' \subset \mathbf{R}^2$. Then A' and A'' are the path components of A, and so $H^0(A;\mathbf{Z}) \approx \mathbf{Z} \oplus \mathbf{Z}$. Since A is connected, in any open neighborhood U of A in $\mathbf{R}^2$, A' and A'' must be in the same path component of U (the path components of U are the same as the components of U because U is locally path connected). It follows that $\bar{H}^0(A;\mathbf{Z}) = \lim_{\to} \{H^0(U;\mathbf{Z})\}$, where U varies over the connected open neighborhoods of A in $\mathbf{R}^2$. Therefore $\bar{H}^0(A;\mathbf{Z}) \approx \mathbf{Z}$ and $i\colon \bar{H}^0(A;\mathbf{Z}) \to H^0(A;\mathbf{Z})$ is not an epimorphism. Thus A is not a taut subspace of $\mathbf{R}^2$ with respect to singular cohomology.

9 LEMMA *Let (A,B) be a pair in X. Then, if two of the three pairs $(B,\varnothing)$, $(A,\varnothing)$, and (A,B) are taut in X, so is the third.*

PROOF This follows from the exactness of the cohomology sequence of a triple, from the fact that a direct limit of exact sequences is exact, and from the five lemma. ∎

Recall (exercise set 1.C) that a normal space X is an absolute neighborhood retract if it has the property that whenever it is imbedded as a closed subset of a normal space, it is a retract of some neighborhood. Also recall that a space X is binormal if $X \times I$ (hence also X) is normal.

10 THEOREM *Any imbedding of an absolute neighborhood retract as a closed subspace of a binormal absolute neighborhood retract is taut.*

PROOF Assume $A \subset X$, where A and X are absolute neighborhood retracts and A is closed in the binormal space X. There is a neighborhood U of A in X such that A is a retract in U. Then $H^*(U) \to H^*(A)$ is an epimorphism, and this implies that

$$i\colon \bar{H}^*(A) \to H^*(A)$$

is an epimorphism.

To show that it is also a monomorphism, let U be an open neighborhood of A in X. There is a closed neighborhood U' of A in U of which A is a retract. Let $r\colon U' \to A$ be a retraction and define a map

$$F\colon (U' \times 0) \cup (A \times I) \cup (U' \times 1) \to U$$

by $F(x,0) = x$ and $F(x,1) = r(x)$ for $x \in U'$ and $F(x,t) = x$ for $x \in A$ and $t \in I$. Because A is closed in X, $(U' \times 0) \cup (A \times I) \cup (U' \times 1)$ is closed in $U' \times I$, the latter being a normal space because it is a closed subset of the normal space $X \times I$. Since U is an open subset of the absolute neighborhood retract X, it follows (see exercise 1.C.4) that U is an absolute neighborhood retract and F can be extended to a map $F'\colon N \to U$, where N is a neighborhood of $(U' \times 0) \cup (A \times I) \cup (U' \times 1)$ in $U' \times I$. N contains a set of the form $V \times I$, where V is a neighborhood of A in U', and $F' \mid V \times I$ is a homotopy from the inclusion map $j\colon V \subset U$ to kr', where $r' = r \mid V\colon V \to A$ and $k\colon A \subset U$. Therefore there is a commutative triangle

$$\begin{array}{ccc} H^*(U) & \xrightarrow{k^*} & H^*(A) \\ & {\scriptstyle j^*}\searrow \quad \swarrow{\scriptstyle r'^*} & \\ & H^*(V) & \end{array}$$

which shows that $\ker k^* \subset \ker j^*$. Thus, if an element in $H^*(U)$ restricts to 0 in $H^*(A)$, it restricts to 0 in $H^*(V)$ for some smaller neighborhood V, hence it represents 0 in $\lim_{\rightarrow} \{H^*(U)\} = \bar{H}^*(A)$. Therefore $i\colon \bar{H}^*(A) \to H^*(A)$ is a monomorphism and A is taut in X. ∎

11 COROLLARY *If A, B, and X are compact polyhedra, any imbedding of (A,B) in X is taut.*

PROOF This follows from the fact (exercise 3.A.1) that a compact polyhedron is an absolute neighborhood retract and from theorem 10 and lemma 9. ∎

One reason for introducing the modules $\bar{H}^q(A,B;\,G)$ is the following result, which asserts that any pair (A,B) in X is taut with respect to the functor $\bar{H}^q$.

12 THEOREM *As U varies over the neighborhoods of A, there is an isomorphism*

$$\lim_{\rightarrow} \{\bar{H}^q(U;G)\} \approx \bar{H}^q(A;G)$$

PROOF Restricting U to the cofinal family of open neighborhoods, we have $\bar{H}^q(U;G) = H^q(U;G)$, and the limit on the left is, by definition, equal to the module on the right. ∎

If (A,B) and (A',B') are pairs in X and (U,V) and (U',V') are respective open neighborhoods, there is a relative Mayer-Vietoris sequence of $\{(U,V), (U',V')\}$. As (U,V) and (U',V') vary over open neighborhoods of (A,B) and (A',B'), respectively, $(U \cup U', V \cup V')$ varies over a cofinal family of neighborhoods of $(A \cup A', B \cup B')$. If (A,B) and (A',B') are closed pairs in a normal space X, it is also true that $(U \cap U', V \cap V')$ varies over a cofinal family of neighborhoods of $(A \cap A', B \cap B')$. Because the direct limit of exact sequences is exact, we obtain the following result, which is another reason for our interest in the modules $\bar{H}^*(A,B)$.

13 THEOREM *If (A,B) and (A',B') are closed pairs in a normal space X, there is an exact relative Mayer-Vietoris sequence (for any coefficient module G)*

$$\cdots \to \bar{H}^q(A \cup A', B \cup B') \to \bar{H}^q(A,B) \oplus \bar{H}^q(A',B') \to \bar{H}^q(A \cap A', B \cap B') \to \cdots \quad ∎$$

Given $u \in H^n(X \times X, X \times X - \delta(X);\, R)$, as (U,V) varies over neighborhoods of (A,B), the homomorphisms

$$\gamma_u\colon H_q(X - V, X - U;\, G) \to H^{n-q}(U,V;\, G)$$

define a homomorphism

$$\lim_{\rightarrow} \{H_q(X - V, X - U; G)\} \to \lim_{\rightarrow} \{H^{n-q}(U,V; G)\}$$

Because singular homology has compact supports, if X is a Hausdorff space the limit on the left is isomorphic to $H_q(X - B, X - A; G)$. Therefore we obtain a natural homomorphism

$$\bar{\gamma}_u\colon H_q(X - B, X - A; G) \to \bar{H}^{n-q}(A,B; G)$$

such that if (U,V) is a neighborhood of (A,B), there is a commutative diagram (all coefficients G)

$$\begin{tikzcd} H_q(X - V, X - U) \arrow[r] \arrow[d, "\gamma_u"'] & H_q(X - B, X - A) \arrow[d, "\bar{\gamma}_u"] \arrow[dr, "\gamma_u"] & \\ H^{n-q}(U,V) \arrow[r] & \bar{H}^{n-q}(A,B) \arrow[r, "i"] & H^{n-q}(A,B) \end{tikzcd}$$

If (A,B) and (A',B') are closed pairs in a normal space X, then $\bar{\gamma}_u$ maps the exact Mayer-Vietoris sequence of the couple of open pairs

$$\{(X - B, X - A), (X - B', X - A')\}$$

into the exact Mayer-Vietoris sequence of theorem 13 in such a way that each square is commutative up to sign.

2 DUALITY IN TOPOLOGICAL MANIFOLDS

This section is devoted to a study of homology properties of topological manifolds. Over a connected manifold as base space there is a fiber-bundle pair called the homology tangent bundle. An orientation class of this bundle gives rise to a duality in the manifold asserting that the cohomology of a compact pair in the manifold is isomorphic to the homology of its complement. This duality theorem is proved by using the orientation class and the slant product to define a natural homomorphism from homology to cohomology. The resulting homomorphism is shown to be an isomorphism by proving it first in euclidean space and then in an arbitrary manifold using the piecing-together technique based on Mayer-Vietoris sequences.

A topological *n-manifold* (*without boundary*) is a paracompact Hausdorff space in which each point has an open neighborhood homeomorphic to $\mathbf{R}^n$ (called a *coordinate neighborhood* in the manifold). Following are some examples of n-manifolds.

1 $\mathbf{R}^n$ and S^n are n-manifolds.

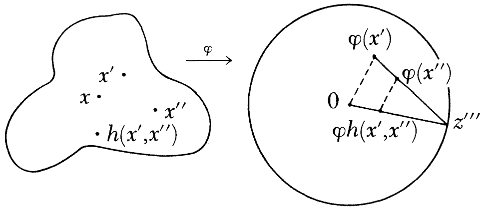
2 An open subset of an n-manifold is an n-manifold.

3 The product of an n-manifold and an m-manifold is an $(n + m)$-manifold.

4 P^n is an n-manifold, $P_n(\mathbf{C})$ a $2n$-manifold, and $P_n(\mathbf{Q})$ a $4n$-manifold for all n. In fact, if X denotes one of these spaces and is coordinatized by homogeneous coordinates $[t_0,t_1, \ldots ,t_n]$, then for each $0 \leq i \leq n$ the subset $A_i \subset X$ of points having ith coordinate 0 is a projective space of dimension $n - 1$ and $X - A_i$ is homeomorphic to $\mathbf{R}$, $\mathbf{R}^2$, or $\mathbf{R}^4$, respectively. Hence, $X - A_i$ is a coordinate neighborhood of X, and X is covered by these $n + 1$ coordinate neighborhoods.

5 LEMMA *In an n-manifold X each point x has an open neighborhood V such that $(V \times X, V \times X - \delta(V))$ is homeomorphic to $V \times (X, X - x)$ by a homeomorphism preserving first coordinates.*

PROOF Let U be a coordinate neighborhood containing x. Without loss of generality, we can suppose that there is a homeomorphism φ: $U \approx \mathbf{R}^n$ such that $\varphi(x) = 0$. Let $D' = \{z \in \mathbf{R}^n \mid \|z\| \leq 2\}$ and $V' = \{z \in \mathbf{R}^n \mid \|z\| < 1\}$ and define $D = \varphi^{-1}(D')$ and $V = \varphi^{-1}(V')$. Then V is an open neighborhood of x contained in the compact set D. If $(x',x'') \in V \times D - \delta(V)$, there is a unique point $z''' \in \mathbf{R}^n$ such that $\|z'''\| = 2$ and $\varphi(x'')$ belongs to the closed segment from $\varphi(x')$ to z'''. If $\varphi(x'') = t\varphi(x') + (1 - t)z'''$, with $t \in I$, let $h(x',x'') \in D - x$ be the point such that $\varphi h(x',x'') = (1 - t)z'''$, as illustrated



D D'

and define $h(x',x') = x$. A homeomorphism

$$\psi: (V \times X, V \times X - \delta(X)) \approx V \times (X, X - x)$$

having the desired properties is defined by

$$\psi(x',x'') = \begin{cases} (x',x'') & x'' \notin D \\ (x', h(x',x'')) & x'' \in D \end{cases} \quad \blacksquare$$

It follows from lemma 5 that if $x' \in V$ then $(X, X - x')$ is homeomorphic to $(X, X - x)$. Hence we obtain the following result.

6 COROLLARY *In a connected n-manifold X the group of homeomorphisms acts transitively; in particular, the topological type of $(X, X - x)$ is independent of x. Furthermore, projection to the first factor p: $X \times X \to X$ is the projection of a fiber-bundle pair $(X \times X, X \times X - \delta(X))$ with fiber pair $(X, X - x)$.* $\blacksquare$

If V is a coordinate neighborhood of x in an n-manifold X, the couple $\{V, X - x\}$ is excisive, and so there is an excision isomorphism

$$H_*(V, V - x; G) \approx H_*(X, X - x; G)$$

Since $H_*(V, V - x; G) \approx H_*(\mathbf{R}^n, \mathbf{R}^n - 0; G)$, it follows that

$$H_q(X, X - x; G) \approx \begin{cases} 0 & q \neq n \\ G & q = n \end{cases}$$

and so the fiber pair $(X, X - x)$ of the fiber-bundle pair of corollary 6 has the same homology as $(\mathbf{R}^n, \mathbf{R}^n - 0)$. For this reason the fiber-bundle pair of corollary 6 will be called the *homology tangent bundle* of X (the tangent bundle itself is an n-plane bundle defined if X is a differentiable manifold and having homology properties isomorphic to those of the homology tangent bundle).

A connected n-manifold X is said to be *orientable* (over R) if its homology tangent bundle is orientable [that is, if there exists an element $U \in H^n(X \times X, X \times X - \delta(X); R)$ such that for all $x \in X$, $U \mid x \times (X, X - x)$ is a generator of $H^n(x \times (X, X - x); R)$]. Such a cohomology class U is called an *orientation* of X. An n-manifold X (which is not assumed to be connected) is said to be *orientable* if each component is orientable, and an *orientation of* X is defined to be a cohomology class $U \in H^n(X \times X, X \times X - \delta(X); R)$ whose restriction to each component is an orientation of that component.

7 EXAMPLE For $\mathbf{R}^n$ the fiber-bundle pair $(\mathbf{R}^n \times \mathbf{R}^n, \mathbf{R}^n \times \mathbf{R}^n - \delta(\mathbf{R}^n))$ is trivial, because the map

$$f(z,z') = (z, z' - z)$$

is a homeomorphism $f\colon (\mathbf{R}^n \times \mathbf{R}^n, \mathbf{R}^n \times \mathbf{R}^n - \delta(\mathbf{R}^n)) \approx \mathbf{R}^n \times (\mathbf{R}^n, \mathbf{R}^n - 0)$ preserving first coordinates. Therefore $\mathbf{R}^n$ is an orientable n-manifold.

The results of Sec. 5.7 dealing with the homology properties of sphere bundles carry over to the homology tangent bundle. We list some of these explicitly.

8 *Two orientations U and U' of a connected manifold X are equal if and only if for some $x_0 \in X$*

$$U \mid x_0 \times (X, X - x_0) = U' \mid x_0 \times (X, X - x_0) \quad \blacksquare$$

9 *Any manifold has a unique orientation over Z_2.* $\blacksquare$

10 *A simply connected manifold is orientable over any R.* $\blacksquare$

11 *An n-manifold X is orientable if and only if there is an open covering $\{V\}$ of X and a compatible family $\{U_V \in H^n(V \times X, V \times X - \delta(V); R)\}$, where U_V corresponds to an orientation of V under the excision isomorphism*

$$H^n(V \times X, V \times X - \delta(V); R) \approx H^n(V \times V, V \times V - \delta(V); R) \quad \blacksquare$$

The duality theorem asserts that if $U \in H^n(X \times X, X \times X - \delta(X); R)$ is an orientation of X, then for any compact pair (A,B) in X, $\bar{\gamma}_U$ is an isomorphism of $H_q(X - B, X - A; G)$ onto $\bar{H}^{n-q}(A,B; G)$. We prove this first for $\mathbf{R}^n$ by a sequence of lemmas.

12 LEMMA *Let $A \subset \mathbf{R}^n$ be homeomorphic to a simplex and let $a_0 \in A$. Then $H_q(\mathbf{R}^n - a_0, \mathbf{R}^n - A; G) = 0$ for all q and G.*

PROOF Regarding $\mathbf{R}^n$ as an open subset of S^n, there is an excision isomorphism $H_q(\mathbf{R}^n - a_0, \mathbf{R}^n - A; G) \approx H_q(S^n - a_0, S^n - A; G)$. Because $S^n - a_0$ is homeomorphic to $\mathbf{R}^n$, $\tilde{H}_q(S^n - a_0; G) = 0$. From lemma 4.7.13 and the universal-coefficient formula, $\tilde{H}_q(S^n - A; G) = 0$. The lemma now follows from exactness of the reduced homology sequence of the pair $(S^n - a_0, S^n - A)$. ∎

13 COROLLARY *If $A \subset \mathbf{R}^n$ is homeomorphic to a simplex and U is an orientation of $\mathbf{R}^n$ over R, then for all q and R modules G*

$$\gamma_U\colon H_q(\mathbf{R}^n, \mathbf{R}^n - A; G) \approx H^{n-q}(A;G)$$

PROOF Let $a_0 \in A$ and consider the diagram (all coefficients G)

$$\begin{array}{ccccccccc}
\cdots \to & H_q(\mathbf{R}^n - a_0, \mathbf{R}^n - A) & \to & H_q(\mathbf{R}^n, \mathbf{R}^n - A) & \to & H_q(\mathbf{R}^n, \mathbf{R}^n - a_0) & \to & H_{q-1}(\mathbf{R}^n - a_0, \mathbf{R}^n - A) & \to \cdots \\
& \downarrow{\scriptstyle \gamma_U} & & \downarrow{\scriptstyle \gamma_U} & & \downarrow{\scriptstyle \gamma_U} & & \downarrow{\scriptstyle \gamma_U} & \\
\cdots \to & H^{n-q}(A,a_0) & \to & H^{n-q}(A) & \to & H^{n-q}(a_0) & \to & H^{n-q+1}(A,a_0) & \to \cdots
\end{array}$$

The rows are exact, and each square either commutes or anticommutes. Since A is contractible, $H^*(A,a_0) = 0$. Using lemma 12, we see that trivially $\gamma_U\colon H_q(\mathbf{R}^n - a_0, \mathbf{R}^n - A) \approx H^{n-q}(A,a_0)$. By the five lemma, to complete the proof we need only verify that $\gamma_U\colon H_q(\mathbf{R}^n, \mathbf{R}^n - a_0) \approx H^{n-q}(a_0)$. Because U is an orientation, $U \mid [a_0 \times (\mathbf{R}^n, \mathbf{R}^n - a_0)] = 1 \times u$, where $u \in H^n(\mathbf{R}^n, \mathbf{R}^n - a_0; R)$ is a generator. By property 6.1.2,

$$\gamma_U(z) = \langle u,z \rangle 1$$

Since u is a generator of $H^n(\mathbf{R}^n, \mathbf{R}^n - a_0; R) \approx \operatorname{Hom}(H_n(\mathbf{R}^n, \mathbf{R}^n - a_0; R), R)$, it follows that the map $z \to \langle u,z \rangle$ of $H_n(\mathbf{R}^n, \mathbf{R}^n - a_0; R)$ to R is an isomorphism; and hence so is $\gamma_U\colon H_n(\mathbf{R}^n, \mathbf{R}^n - a_0; R) \approx H^0(a_0;R)$. If $q \neq n$, it is trivially true that $\gamma_U\colon H_q(\mathbf{R}^n, \mathbf{R}^n - a_0; R) \approx H^{n-q}(a_0;R)$, since both modules are trivial. ∎

14 THEOREM *If U is an orientation of $\mathbf{R}^n$ over R and (A,B) is a compact polyhedral pair in $\mathbf{R}^n$, then for all q and all R modules G there is an isomorphism*

$$\gamma_U\colon H_q(\mathbf{R}^n - B, \mathbf{R}^n - A; G) \approx H^{n-q}(A,B; G)$$

PROOF Because of the naturality properties of γ_U, it suffices to prove this for the case where B is empty. The theorem follows for A from corollary 13 by induction on the number of simplexes in a triangulation of A, using Mayer-Vietoris sequences and the five lemma. ∎

15 COROLLARY *If U is an orientation of $\mathbf{R}^n$ over R and (A,B) is a compact pair in $\mathbf{R}^n$, then for all q and R modules G there is an isomorphism*

$$\bar{\gamma}_U\colon H_q(\mathbf{R}^n - B, \mathbf{R}^n - A; G) \approx \bar{H}^{n-q}(A,B; G)$$

PROOF Since the family of compact polyhedral pairs is cofinal in the family of all neighborhoods of a compact pair (A,B) in $\mathbf{R}^n$, the corollary follows from theorem 14 by taking direct limits. ∎

Because of the commutativity of the triangle

$$\begin{array}{ccc} & H_q(\mathbf{R}^n - B, \mathbf{R}^n - A; G) & \\ {}^{\bar\gamma_U}\swarrow & & \searrow{}^{\gamma_U} \\ \bar H^{n-q}(A,B; G) & \xrightarrow{i} & H^{n-q}(A,B; G) \end{array}$$

it follows from theorem 14 and corollary 15 that any imbedding of a compact polyhedral pair in $\mathbf{R}^n$ is taut (which is also a consequence of corollary 6.1.11).

As an immediate result of corollary 15, we obtain the following *Alexander duality theorem.*

16 THEOREM *If A is a compact subset of $\mathbf{R}^n$, then for all q and R modules G*

$$\tilde H_q(\mathbf{R}^n - A; G) \approx \bar{\tilde H}{}^{n-q-1}(A;G)$$

PROOF Because $\tilde H_*(\mathbf{R}^n;G) = 0$, there is an isomorphism

$$\partial_*: H_{q+1}(\mathbf{R}^n, \mathbf{R}^n - A; G) \approx \tilde H_q(\mathbf{R}^n - A; G)$$

The result is obtained by composing the inverse of this isomorphism with the isomorphism of corollary 15. ∎

For general orientable manifolds there is the following *duality theorem.*

17 THEOREM *Let U be an orientation over R of an n-manifold X and let (A,B) be a compact pair in X. Then for all q and R modules G there is an isomorphism*

$$\bar\gamma_U: H_q(X - B, X - A; G) \approx \bar H^{n-q}(A,B; G)$$

PROOF Because of the naturality properties of $\bar\gamma_U$, it suffices to prove the theorem for the case where B is empty. If A is contained in some coordinate neighborhood V of X and $U' = U \mid (V \times V, V \times V - \delta(V))$ is the induced orientation of V, there is a commutative triangle (all coefficients G)

$$\begin{array}{ccc} H_q(V, V - A) & \underset{\approx}{\to} & H_q(X, X - A) \\ {}^{\bar\gamma_{U'}}\searrow & & \swarrow{}^{\bar\gamma_U} \\ & \bar H^{n-q}(A) & \end{array}$$

By corollary 15, $\bar\gamma_{U'}$ is an isomorphism, hence $\bar\gamma_U$ is also an isomorphism. The result for arbitrary compact A follows by induction on the finite number of coordinate neighborhoods needed to cover A, using naturality of $\bar\gamma_U$, the usual Mayer-Vietoris technique, and the five lemma. ∎

In case X is compact, by applying theorem 17 to the pair $(X, \varnothing)$ and observing that $i: \bar H^q(X;G) \approx H^q(X;G)$, we obtain the following *Poincaré duality theorem.*

18 THEOREM *If U is an orientation over R of a compact n-manifold X, then for all q and R modules G there is an isomorphism*

$$\gamma_U\colon H_q(X;G) \approx H^{n-q}(X;G) \quad \blacksquare$$

A pair (X,A) is called a *relative n-manifold* if X is a Hausdorff space, A is closed in X (A may be empty), and $X - A$ is an n-manifold. For relative manifolds there is the following *Lefschetz duality theorem.*

19 THEOREM *Let (X,A) be a compact relative n-manifold such that $X - A$ is orientable over R. For all q and R modules G there is an isomorphism*

$$H_q(X - A;\, G) \approx \bar{H}^{n-q}(X,A;\, G)$$

PROOF Let $\{N\}$ be the family of closed neighborhoods of A directed downward by inclusion. There are isomorphisms

$$\lim_{\rightarrow} \{H_q(X - N;\, G)\} \approx H_q(X - A;\, G)$$
$$\lim_{\rightarrow} \{\bar{H}^{n-q}(X,N;\, G)\} \approx \bar{H}^{n-q}(X,A;\, G)$$

the first because singular homology has compact supports and the second as a consequence of theorem 6.1.12. Let V be an open neighborhood of A with $\bar{V}$ contained in the interior of N and let U be an orientation of $X - A$ over R. By theorem 17 and standard excision properties, there are isomorphisms (all coefficients G)

$$\begin{array}{ccc} H_q(X - N) & \xrightarrow{\approx} & H_q((X - A) - (N - V),\, (X - A) - (X - V)) \\ & & \approx \big\downarrow \bar{\gamma}_U \\ \bar{H}^{n-q}(X,N) & \xrightarrow{\approx} & \bar{H}^{n-q}(X - V,\, N - V) \end{array}$$

which yield the result on passing to the limit. $\blacksquare$

An *n-manifold X with boundary $\dot{X}$* is a paracompact Hausdorff space such that $(X,\dot{X})$ is a relative n-manifold and every point $x \in \dot{X}$ has a neighborhood V such that $(V,\, V \cap \dot{X})$ is homeomorphic to $\mathbf{R}^{n-1} \times (I,0)$. Since $\dot{X}$ may be empty, the concept of manifold with boundary encompasses that of manifold without boundary.

If X is an n-manifold with boundary $\dot{X}$, then $\dot{X}$ has neighborhoods N such that $(N,\dot{X})$ is homeomorphic to $\dot{X} \times (I,0)$.[1] Such a neighborhood N is called a *collaring* of $\dot{X}$, and its interior is called an *open collaring* of $\dot{X}$. (In case $\dot{X}$ is compact, any neighborhood of $\dot{X}$ contains a collaring of $\dot{X}$.) Because of the existence of such collarings, $X - \dot{X}$ is a weak deformation retract of X, and the pair $((X - \dot{X}) \times (X - \dot{X}),\, (X - \dot{X}) \times (X - \dot{X}) - \delta(X - \dot{X}))$ is a weak deformation retract of $(X \times X,\, X \times X - \delta(X))$.

An n-manifold X with boundary $\dot{X}$ is said to be *orientable* over R if $X - \dot{X}$ is orientable over R. An *orientation* over R of X is a class

[1] See M. Brown, Locally flat imbeddings of topological manifolds, *Annals of Mathematics*, vol. 75, pp. 331–341, 1962.

$U \in H^n(X \times X, X \times X - \delta(X); R)$ whose restriction to $((X - \dot{X}) \times (X - \dot{X}), (X - \dot{X}) \times (X - \dot{X}) - \delta(X - \dot{X}))$ is an orientation of $X - \dot{X}$ over R. For manifolds with boundary the *Lefschetz duality theorem* takes the following form.

20 THEOREM *Let X be a compact n-manifold with boundary $\dot{X}$ and orientation U over R. For all q and R modules G there are isomorphisms (where j: $X - \dot{X} \subset X$)*

$$H_q(X;G) \underset{\approx}{\overset{j_*}{\longleftarrow}} H_q(X - \dot{X}; G) \underset{\approx}{\overset{\gamma_U}{\longrightarrow}} H^{n-q}(X,\dot{X}; G)$$

$$H_q(X,\dot{X}; G) \underset{\approx}{\overset{\gamma_U}{\longrightarrow}} H^{n-q}(X - \dot{X}; G) \underset{\approx}{\overset{j^*}{\longleftarrow}} H^{n-q}(X;G)$$

PROOF Because j is a homotopy equivalence, j_* and j^* are isomorphisms. Let N be a collaring of $\dot{X}$ with interior $\mathring{N}$. Let U' be the orientation of $X - \dot{X}$ obtained by restricting U. In the following commutative diagram each horizonal map is induced by inclusion and is an isomorphism because it is an excision (labelled e) or a homotopy equivalence (labelled h) (all coefficients G):

$$\begin{tikzcd}
H_q(X - \dot{X}) \arrow[d, "\gamma_U"'] & H_q(X - N) \arrow[l, "h", "\approx"'] \arrow[r, "e", "\approx"'] \arrow[d, "\gamma_U"'] & H_q((X - \dot{X}) - (N - \mathring{N}), (X - \dot{X}) - (X - \mathring{N})) \arrow[d, "\gamma_{U'}"] \\
H^{n-q}(X,\dot{X}) & H^{n-q}(X,N) \arrow[l, "h", "\approx"'] \arrow[r, "e", "\approx"'] & H^{n-q}(X - \mathring{N}, N - \mathring{N})
\end{tikzcd}$$

Because $(X - \mathring{N}, N - \mathring{N})$ has arbitrarily small neighborhoods of which it is a deformation retract i: $\bar{H}^{n-q}(X - \mathring{N}, N - \mathring{N}) \approx H^{n-q}(X - \mathring{N}, N - \mathring{N})$, and it follows from theorem 17 that the right-hand vertical map is an isomorphism (because it corresponds to the isomorphism $\bar{\gamma}_{U'}$). Therefore the left-hand vertical map is also an isomorphism proving the first part of the theorem.

Similarly, there is a commutative diagram

$$\begin{tikzcd}
H_q(X,\dot{X}) \arrow[r, "h", "\approx"'] \arrow[d, "\gamma_U"'] & H_q(X,\mathring{N}) \arrow[d, "\gamma_U"'] & H_q(X - \dot{X}, (X - \dot{X}) - (X - \mathring{N})) \arrow[l, "e", "\approx"'] \arrow[d, "\gamma_{U'}"] \\
H^{n-q}(X - \dot{X}) \arrow[r, "h", "\approx"'] & H^{n-q}(X - \mathring{N}) & H^{n-q}(X - \mathring{N}) \arrow[l, "e", "\approx"']
\end{tikzcd}$$

Because $X - \mathring{N}$ has arbitrarily small neighborhoods of which it is a deformation retract, it follows from theorem 17 that the right-hand vertical map is an isomorphism. Therefore the left-hand vertical map is also an isomorphism, proving the second part of the theorem. ■

From the isomorphisms of theorem 20 and the universal-coefficient theorem for homology, we obtain a short exact sequence

$$0 \to H^q(X;R) \otimes G \xrightarrow{\mu} H^q(X;G) \to H^{q+1}(X;R) * G \to 0$$

and a similar short exact sequence for $H^q(X,\dot{X}; G)$. Since this is so for every R module G, from theorem 5.5.13 we have the following result.

21 COROLLARY *If X is a compact n-manifold with boundary $\dot{X}$ orientable over R, then $H_*(X;R)$ and $H_*(X,\dot{X}; R)$ are finitely generated.* ■

Later in the chapter (see theorem 6.9.11) we shall prove that corollary 21 is also valid for nonorientable manifolds.

3 THE FUNDAMENTAL CLASS OF A MANIFOLD

In view of the importance of the concept of orientability of manifolds, we shall now investigate some equivalent formulations. We shall show that a compact connected n-manifold is orientable if and only if its n-dimensional homology module is nonzero. In fact, any orientation class of the manifold will be shown to correspond to a generator of the n-dimensional homology module. Moreover, if z is the element of H_n corresponding to the orientation, then the cap product of z and a cohomology class defines a homomorphism which equals, up to sign, the inverse of the duality isomorphism. The methods in this section rely heavily on the technique of piecing together homology classes,[1] analogous to the piecing together of cohomology classes in lemma 5.7.16.

Let X be a space, X' a subspace of X, and $\mathcal{A} = \{A\}$ a collection of subsets of $X - X'$. A *compatible* $\mathcal{A}$ *family* is a family $\{z_A \in H_q(X, X - A; G)\}$ (for some fixed q and G) indexed by $\mathcal{A}$ such that if $A, A' \in \mathcal{A}$, then z_A and $z_{A'}$ map to the same element of $H_q(X, X - A \cap A'; G)$ under the homomorphisms

$$H_q(X, X - A; G) \to H_q(X, X - A \cap A'; G) \leftarrow H_q(X, X - A'; G)$$

The compatible $\mathcal{A}$ families form a module with respect to componentwise operations that will be denoted by $H_q^{\mathcal{A}}(X,X'; G)$. For the collection $\mathcal{A}$ of all compact subsets of $X - X'$ we use $H_q{}^c(X,X'; G)$ to denote the corresponding module.

We are interested in the module $H_n{}^c(X,\dot{X}; R)$ for an n-manifold X with boundary $\dot{X}$. The following lemma is important in this connection.

1 LEMMA *Let X be an n-manifold with boundary $\dot{X}$ and let A be a compact subset of $X - \dot{X}$. For all R modules G*

$$H_q(X, X - A; G) = 0 \qquad q > n$$

PROOF Assume first that A is contained in some coordinate neighborhood V in $X - \dot{X}$. By excision, $H_q(V, V - A) \approx H_q(X, X - A)$, and since V is homeomorphic to $\mathbf{R}^n$, we can use corollary 6.2.15 to obtain

$$H_q(V, V - A) \approx \bar{H}^{n-q}(A) = 0 \qquad q > n$$

For arbitrary compact A the result follows by induction on the number of coordinate neighborhoods needed to cover A, using Mayer-Vietoris sequences. ■

In an n-manifold X with boundary $\dot{X}$ a *small cell* in $X - \dot{X}$ is defined to be a compact subset A having an open neighborhood $V \subset X - \dot{X}$ such that

[1] This technique can be found in H. Cartan, Méthodes modernes en topologie algébrique, *Commentarii Mathematici Helvetici*, vol. 18, pp. 1–15, 1945.

(V,A) is homeomorphic to $(\mathbf{R}^n,E^n)$. Every point of $X - \dot{X}$ has arbitrarily small neighborhoods which are small cells. If A and V are as above, there is an excision isomorphism

$$H_q(X, X - A; G) \approx H_q(V, V - A; G) \approx \begin{cases} 0 & q \neq n \\ G & q = n \end{cases}$$

If $x_0 \in A$, then the inclusion map induces isomorphisms

$$H_q(X, X - A; G) \approx H_q(X, X - x_0; G)$$

We use $H_q{}^{sc}(X,\dot{X}; G)$ to denote the module of compatible $\mathcal{C}$ families, where $\mathcal{C}$ consists of the collection of small cells of $X - \dot{X}$. Since the collection of small cells is contained in the collection of compact subsets of $X - \dot{X}$, there is a natural homomorphism

$$H_q{}^c(X,\dot{X}; G) \to H_q{}^{sc}(X,\dot{X}; G)$$

which assigns to a compatible family $\{z_A\}$ indexed by all compact A the compatible subfamily of elements indexed by small cells.

2 LEMMA *Let X be an n-manifold with boundary $\dot{X}$. Then, for all G*

$$H_n{}^c(X,\dot{X}; G) \approx H_n{}^{sc}(X,\dot{X}; G)$$

PROOF For each positive integer i let $\mathcal{C}_i$ be the collection of compact subsets of $X - \dot{X}$ contained in the union of i small cells. Then $\mathcal{C}_i \subset \mathcal{C}_{i+1}$ and $\bigcup \mathcal{C}_i$ is the collection of all compact subsets of $X - \dot{X}$. There are homomorphisms

$$\cdots \to H_n^{\mathcal{C}_{i+1}} \to H_n^{\mathcal{C}_i} \to \cdots \to H_n^{\mathcal{C}_1} \to H_n{}^{sc}$$

and an isomorphism $H_n{}^c \approx \lim_{\leftarrow} \{H_n^{\mathcal{C}_i}\}$.

Since every element of $\mathcal{C}_1$ is contained in some small cell, it is obvious that $H_n^{\mathcal{C}_1} \approx H_n{}^{sc}$. By the usual Mayer-Vietoris technique and lemma 1, it follows that for any $i \geq 1$ $H_n^{\mathcal{C}_{i+1}} \approx H_n^{\mathcal{C}_i}$. Combining these isomorphisms yields the result. ∎

This gives the following important result.

3 THEOREM *Let X be an n-manifold with boundary $\dot{X}$ and let*

$$\{z_A\} \in H_n{}^c(X,\dot{X}; G)$$

(*a*) *$\{z_A\} = 0$ if and only if $z_x = 0$ for all $x \in X - \dot{X}$.*

(*b*) *If X is connected, $\{z_A\} = 0$ if and only if $z_x = 0$ for some $x \in X - \dot{X}$.*

PROOF (*a*) follows from lemma 2 and the observation that if A is a small cell and $x \in A$, then

$$H_n(X, X - A; G) \approx H_n(X, X - x; G)$$

and so $z_A = 0$ if and only if $z_x = 0$.

To prove (*b*), assume $z_{x_0} = 0$ for some $x_0 \in X - \dot{X}$. Because X is connected, so is its weak deformation retract $X - \dot{X}$. This implies that if

$x \in X - \dot{X}$, there is a finite sequence of small cells $A_1, \ldots, A_m$ in $X - \dot{X}$ such that $x_0 \in A_1$ and $x \in A_m$, and A_i meets A_{i+1} for $1 \leq i < m$. Choose a point $x_i \in A_i \cap A_{i+1}$ for $1 \leq i < m$. There are isomorphisms

$$H_n(X, X - x_0) \xleftarrow{\approx} H_n(X, X - A_1) \xrightarrow{\approx} H_n(X, X - x_1) \xleftarrow{\approx} \cdots$$
$$\xleftarrow{\approx} H_n(X, X - A_m) \xrightarrow{\approx} H_n(X, X - x)$$

from which it follows that if $z_{x_0} = 0$, then $z_x = 0$. Since this is so for all $x \in X - \dot{X}$, the result follows from (a). ∎

If X is an n-manifold with boundary $\dot{X}$, a *fundamental family of* X over R is an element $\{z_A\} \in H_n{}^c(X,\dot{X}; R)$ such that for all $x \in X - \dot{X}$, z_x is a generator of $H_n(X, X - x; R)$. The relation between fundamental families and orientations is made precise in the next result.

4 THEOREM *Let X be an n-manifold with boundary $\dot{X}$. There is a one-to-one correspondence between orientations U (over R) of X and fundamental families $\{z_A\}$ (over R) of X such that U and $\{z_A\}$ correspond if and only if $\gamma_U(z_A) = 1 \in H^0(A;R)$ for all compact A in $X - \dot{X}$.*

PROOF If U is an orientation of X, let U' be the induced orientation of $X - \dot{X}$. For any compact $A \subset X - \dot{X}$ we have the commutative diagram (all coefficients R)

$$\begin{tikzcd}
H_n(X,X - A) \arrow[d, "\gamma_U"'] & H_n(X - \dot{X}, (X - \dot{X}) - A) \arrow[l, "j_*"', "\approx"] \arrow[dl, "\gamma_{U'}"'] \arrow[d, "\bar{\gamma}_{U'}"] \\
H^0(A) & \bar{H}^0(A) \arrow[l, "i"']
\end{tikzcd}$$

By theorem 6.2.17, the right-hand vertical map is an isomorphism, and since $1 \in H^0(A)$ is the image of $1 \in \bar{H}^0(A)$, there is a unique $z_A \in H_n(X, X - A)$ such that $\bar{\gamma}_{U'} j_*{}^{-1}(z_A) = 1 \in \bar{H}^0(A)$. Because of the uniqueness of z_A and the naturality of γ_U and $\bar{\gamma}_{U'}$, the collection $\{z_A\}$ is a compatible family. From the commutativity of the above diagram, $\gamma_U(z_A) = 1 \in H^0(A)$ for all compact A in $X - \dot{X}$. Hence we need only verify that $\{z_A\}$ is a fundamental family. In case $A = x$, it follows from the commutativity of the above square and the fact that i: $\bar{H}^0(x) \approx H^0(x)$ that γ_U: $H_n(X, X - x) \approx H^0(x)$. Therefore $z_x = \gamma_U{}^{-1}(1)$ is a generator of $H_n(X, X - x)$. Hence $\{z_A\}$ is a fundamental family with the desired property, and the collection $\{z_x\}_{x \in X - \dot{X}}$ (and hence, by theorem 3a, $\{z_A\}$) is uniquely characterized by the property $\gamma_U(z_x) = 1 \in H^0(x)$.

Conversely, given a fundamental family $\{z_A\}$, let V be any open subset of $X - \dot{X}$ homeomorphic to $\mathbf{R}^n$. If $x_0 \in V$, then $H^*(V;R) \approx H^*(x_0;R)$, which implies that

$$H^*(V \times X, V \times X - \delta(V); R) \approx H^*(x_0 \times (X, X - x_0); R)$$

If $u \in H^n(V \times X, V \times X - \delta(V); R)$, it follows from the Künneth formula for cohomology (theorem 5.6.1) that $u \mid x_0 \times (X, X - x_0) = 1 \times u'$ for a unique $u' \in H^n(X, X - x_0; R) \approx \operatorname{Hom}(H_n(X, X - x_0; R), R)$. By property 6.1.2,

$$[u \mid x_0 \times (X, X - x_0)]/z_{x_0} = \langle u', z_{x_0} \rangle 1$$

Since z_{x_0} is a generator of $H_n(X, X - x_0; R)$, $\langle u', z_{x_0} \rangle$ completely determines u'. Therefore there is a unique element $U \in H^n(V \times X, V \times X - \delta(V); R)$ such that $[U \mid x_0 \times (X, X - x_0)]/z_{x_0} = 1 \in H^0(x_0;R)$.

We now show that for any $x \in V$, $[U \mid x \times (X, X - x)]/z_x = 1 \in H^0(x;R)$. If x and x' belong to a small cell $A \subset V$, then z_A maps to z_x and to $z_{x'}$. Therefore $[U \mid A \times (X, X - A)]/z_A \in H^0(A;R)$ maps to $[U \mid x \times (X, X - x)]/z_x$ and to $[U \mid x' \times (X, X - x')]/z_{x'}$ by naturality of γ_U. Since $H^0(A;R) \approx H^0(x;R)$ and $H^0(A;R) \approx H^0(x';R)$, it follows that both $[U \mid x \times (X, X - x)]/z_x = 1 \in H^0(x;R)$ and $[U \mid x' \times (X, X - x')]/z_{x'} = 1 \in H^0(x';R)$ or neither equation is true. Hence the set of $x \in V$ for which $[U \mid x \times (X, X - x)]/z_x = 1 \in H^0(x;R)$ is open and its complement in V is open. Since V is connected and $[U \mid x_0 \times (X, X - x_0)]/z_{x_0} = 1$, it follows that $[U \mid x \times (X, X - x)]/z_x = 1$ for all $x \in V$.

This means that U is an orientation of V, and if U' is a similarly defined orientation for another coordinate neighborhood V' in $X - \dot{X}$, then for any $x \in V \cap V'$, $U \mid x \times (X, X - x) = U' \mid x \times (X, X - x)$. This implies that U and U' induce the same orientation of $V \cap V'$. Hence the collection $\{U_V\}$ for coordinate neighborhoods V in $X - \dot{X}$ is compatible. Therefore there is an orientation U of X such that $U \mid (V \times X, V \times X - \delta(V)) = U_V$. From the construction of U_V we see that $\gamma_U(z_x) = 1 \in H^0(x;R)$ for all $x \in X - \dot{X}$. By the first half of the proof, there is a fundamental family $\{z'_A\}$ such that $\gamma_U(z'_A) = 1 \in H^0(A;R)$. Then $z'_x = z_x$ for all $x \in X - \dot{X}$, and by theorem 3*a*, $z'_A = z_A$ for all compact $A \subset X - \dot{X}$. Therefore $\gamma_U(z_A) = 1 \in H^0(A;R)$ for all A, proving that every fundamental family $\{z_A\}$ corresponds to some orientation U.

The orientation U is uniquely characterized by the fundamental family $\{z_A\}$, for if U and U' are two orientations of X such that $\gamma_U(z_x) = \gamma_{U'}(z_x)$ for all $x \in X - \dot{X}$, then $U \mid x \times (X, X - x) = U' \mid x \times (X, X - x)$ for all $x \in X - \dot{X}$. Therefore, by lemma 5.7.13, $U = U'$. ∎

This last result gives the following useful characterization of orientability for connected manifolds.

5 THEOREM *Let X be a connected n-manifold with boundary $\dot{X}$. If $H_n{}^c(X,\dot{X}; R) \neq 0$, then $H_n{}^c(X,\dot{X}; R) \approx R$ and any generator is a fundamental family of X.*

PROOF From theorem 3*b* it follows that, given $x_0 \in X - \dot{X}$, the homomorphism

$$H_n{}^c(X,\dot{X}; R) \to H_n(X, X - x_0; R)$$

sending $\{z_A\}$ to z_{x_0} is a monomorphism. Since $H_n(X, X - x_0; R) \approx R$, either $H_n{}^c(X,\dot{X}; R) = 0$ or $H_n{}^c(X,\dot{X}; R) \approx R$. Assume $H_n{}^c(X,\dot{X}; R) \approx R$ and let $\{z_A\}$ be a generator of $H_n{}^c(X,\dot{X}; R)$. Assume that for some $x \in X - \dot{X}$, z_x is not a generator of $H_n(X, X - x; R)$. There is then a noninvertible element $r \in R$ such that $z_x = rz'_x$ for some $z'_x \in H_n(X, X - x; R)$. It follows that for any small cell A containing x, $z_A = rz'_A$ for some $z'_A \in H_n(X, X - A; R)$. Because X

is connected, it follows, as in the proof of theorem 3b, that for any small cell A in $X - \dot{X}$, $z_A = rz'_A$ for some $z'_A \in H_n(X, X - A; R)$. If A' is a small cell in A, then rz'_A maps to $rz'_{A'}$ in $H_n(X, X - A'; R)$. Because $H_n(X, X - A'; R)$ is torsion free, by lemma 1, z'_A maps to $z'_{A'}$. Therefore $\{z'_A\} \in H_n{}^{sc}(X,\dot{X}; R)$. By lemma 2, it follows that the original element $\{z_A\} \in H_n{}^c(X,\dot{X}; R)$ is divisible by the element $r \in R$. Since r is not invertible, this contradicts the hypothesis that $\{z_A\}$ is a generator of $H_n{}^c(X,\dot{X}; R)$. ■

6 COROLLARY *If X is a connected n-manifold with boundary $\dot{X}$, then X is orientable over R if and only if $H_n{}^c(X,\dot{X}; R) \neq 0$.*

PROOF This is immediate from theorems 4 and 5. ■

We now specialize to the case of a compact manifold.

7 LEMMA *If X is a compact n-manifold with boundary $\dot{X}$, there is an isomorphism*

$$H_n(X,\dot{X}; G) \approx H_n{}^c(X,\dot{X}; G)$$

sending $z \in H_n(X,\dot{X}; G)$ to $\{z_A = \text{image of } z \text{ in } H_n(X, X - A; G)\}$.

PROOF Let V be an open collaring of $\dot{X}$ and let $B = X - V$. Then B is compact and there is a homomorphism

$$H_n{}^c(X,\dot{X}; G) \to H_n(X, X - B; G)$$

sending $\{z_A\}$ to z_B. Since $X - B = V$ and $(X,\dot{X}) \subset (X,V)$ is a homotopy equivalence, the composite

$$H_n(X,\dot{X}; G) \to H_n{}^c(X,\dot{X}; G) \to H_n(X, X - B; G)$$

is an isomorphism. To complete the proof we need only show that the right-hand map is a monomorphism. Assume that $\{z_A\}$ is a compatible family such that $z_B = 0$ and let A be any compact set in $X - \dot{X}$. There is then an open collaring V' of $\dot{X}$ such that $V' \subset V$ and V' is disjoint from A. Let $B' = X - V'$. Then A, $B \subset B'$, and we have homomorphisms (all coefficients G)

$$H_n(X, X - A) \leftarrow H_n(X, X - B') \xrightarrow{\approx} H_n(X, X - B)$$

the second map being an isomorphism because $(X,V') \subset (X,V)$ is a homotopy equivalence. Since $z_B = 0$, $z_{B'} = 0$ and $z_A = 0$. Therefore $\{z_A\} = 0$ in $H_n{}^c(X,\dot{X}; G)$. ■

8 COROLLARY *A compact connected n-manifold X with boundary $\dot{X}$ is orientable over R if and only if $H_n(X,\dot{X}; R) \neq 0$.*

PROOF This is immediate from corollary 6 and lemma 7. ■

If X is a compact n-manifold with boundary $\dot{X}$, a *fundamental class* over R of X is an element $z \in H_n(X,\dot{X}; R)$ whose image in $H_n{}^c(X,\dot{X}; R)$ under the isomorphism of lemma 7 is a fundamental family [that is, for every $x \in X - \dot{X}$ the image of z in $H_n(X, X - x; R)$ is a generator of the latter].

9 THEOREM *If X is a compact n-manifold with boundary $\dot{X}$, there is a one-to-one correspondence between orientations U over R and fundamental classes z over R such that U corresponds to z if and only if $\gamma_U(z) = 1 \in H^0(X;R)$.*

PROOF This follows from theorem 4 and lemma 7 on observing that an element $v \in H^0(X;R)$ equals 1 if and only if $v \mid x = 1 \in H^0(x;R)$ for all $x \in X - \dot{X}$. ∎

10 COROLLARY *If X is a compact n-manifold with boundary $\dot{X}$, then if X is orientable, so is $\dot{X}$, and any fundamental class of X maps to a fundamental class of $\dot{X}$ under the connecting homomorphism*

$$\partial_*: H_n(X,\dot{X}; R) \to H_{n-1}(\dot{X};R)$$

PROOF Let N be a collaring of $\dot{X}$ with interior $\mathring{N}$. Then N is an n-manifold with boundary $\dot{X} \cup (N - \mathring{N})$, and there is a commutative diagram (all coefficients R)

$$\begin{array}{ccccc}
H_n(X,\dot{X}) & & \xrightarrow{i_*} & & H_n(X, \dot{X} \cup (X - \mathring{N})) \\
\partial_* \downarrow & & & & j_* \uparrow \approx \\
H_{n-1}(\dot{X}) & \underset{\approx}{\xrightarrow{k_*}} & H_{n-1}(\dot{X} \cup (N - \mathring{N}), N - \mathring{N}) & \xleftarrow{\partial_*} & H_n(N, \dot{X} \cup (N - \mathring{N}))
\end{array}$$

It is clear from the definition of fundamental class that if $z \in H_n(X,\dot{X})$ is a fundamental class of X, then $j_*{}^{-1}i_* z = z'$ is a fundamental class of N. Because N is homeomorphic to $\dot{X} \times I$ in such a way that $\dot{X}$ and $N - \mathring{N}$ correspond to $\dot{X} \times 0$ and $\dot{X} \times 1$, respectively, the Künneth formula implies

$$H_n(N, \dot{X} \cup (N - \mathring{N})) \approx H_{n-1}(\dot{X}) \otimes H_1(I,\dot{I})$$

Let $w \in H_1(I,\dot{I})$ be a generator and let $\{\dot{X}_j\}$ be the components of $\dot{X}$. Then z' corresponds to $\sum z_j' \times w$ for some $z_j' \in H_{n-1}(\dot{X}_j)$, and $k_*{}^{-1}\partial_* z' = \pm\sum z_j'$. Hence $\partial_* z = \pm\sum z_j'$, and since z is a fundamental class of X, $z_j' \times w$ corresponds to a fundamental class of $\dot{X}_j \times I$. Therefore z_j' is nonzero and is a generator of $H_{n-1}(\dot{X}_j)$. Then z_j' is a fundamental class of $\dot{X}_j$, whence $\pm\sum z_j' = \partial_* z$ is a fundamental class of $\dot{X}$. ∎

We are now heading toward a proof that cap product with a fundamental class is an isomorphism which, up to sign, is inverse to the duality isomorphism in a compact manifold. First we need a lemma.

11 LEMMA *Let X be a compact orientable n-manifold with boundary $\dot{X}$ and let p_1, p_2: $X \times X \to X$ be the projections. Given*

$$u \in H^q(X \times X, X \times X - \delta(X); R),\ z \in H_m(X \times X, X \times X - \delta(X); G),$$

and $v \in H^r(X;G)$, then

$$\begin{array}{lll}
p_{1*}(u \frown z) = p_{2*}(u \frown z) & \text{in} & H_{m-q}(X;G) \\
u \smile p_1^* v = u \smile p_2^* v & \text{in} & H^{q+r}(X \times X, X \times X - \delta(X); G)
\end{array}$$

PROOF Let T: $(X \times X, X \times X - \delta(X)) \to (X \times X, X \times X - \delta(X))$ be the